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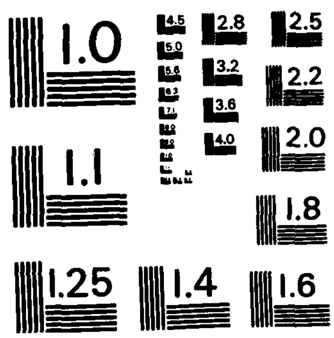
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A GENERAL MODEL FOR THE HOMOGENEOUS CASE OF THE CONTINUOUS RESPONSE

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A GENERAL MODEL FOR THE HOMOGENEOUS CASE OF THE CONTINUOUS RESPONSE

ABSTRACT

A general model for the homogeneous case of the continuous response is proposed. The model is an expansion and generalization of the one proposed by the author in 1974, in which the open response situation is dealt with. In this generalized model, we deal with the closed response situation, and it includes the model for the open response situation as a special case. It also includes models for the open/closed and the closed/open response situations as two special cases. The distinction among the four response situations depends upon the probability assigned to each of the two extreme values of the continuous response, i.e., the probability zero corresponds to the word "open," and non-zero to "closed." Some examples are given.

The research was conducted at the principal investigator's laboratory, 405 Austin Peay Hall, Department of Psychology, University of Tennessee Knoxville, Tennessee. Those who worked for her as assistants include Paul S. Changas, Vicki R. Newton, Donald Reece Dana, Elizabeth Morgan, and Esther Brunton.

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I Introduction

Latent trait theory has been expanded and generalized in the past couple of decades to include varieties of different situations. Although it started from the simplest case, i.e., the dichomotomous response level (e.g., Lord, 1952; Lazarsfeld, 1959; Rasch, 1960; Birnbaum, 1968), now it includes varieties of cases such as the graded response level (e.g., Samejima, 1969, 1972), the nominal response level (e.g., Bock, 1972; Samejima, 1972, 1981) and the continuous response level (e.g., Samejima, 1973, 1974). This expansion and generalization of the theory has tremendously broadened the area of application of latent trait theory and produced varieties of its methods. Researchers can select suitable situations and appropriate models for their research and data, depending, mainly, upon the manner with which the subject's response is given and scored. Almost all the other areas of psychology, in addition to mental and social attitude measurements which the latent trait model originally stemmed from, are now included in the area of application. Marketing and medicine (cf. Roche, Wainer and Thissen, 1975) are two examples of the new areas which benefit from the expanded theory.

The distinction between the continuous response level and the discrete response level has been made by the fact that in the former situation the conditional distribution of the item score, given the latent trait, is continuous, whereas in the latter it is discrete. Samejima has proposed the homogeneous case of the continuous response model (Samejima, 1973), in which continuous item response distributions are solely observed. In her paper, she distinguished two situations, i.e., the open response situation and the closed response situation. The general model was proposed, basically, for the open response situation.

This clear twofold categorization may not always hold, however, for psychometrics must deal with much complicated psychological reality, including situations in which the item response distribution is partly discrete and partly continuous. Take a cognitive process, for example. It is rather customary that cognitive psychologists take the response latency as a measure of the subject's performance. This is justified when the required cognitive task is relatively uncomplicated, and the response latency can be interpreted to have a straightforward relationship with the cognitive process of interest. Suppose that in our experiment a relatively simple problem is presented for the subject to solve, and the subject is given a sufficiently long time for the task. In such a case we can apply a model which belongs to the general continuous response model (Samejima, 1973), using the subject's response latency as the reversed continuous item score. If we set a time limit for the problem solving in order to facilitate the experiment, however, we may not be able to adopt such a model, for our procedure assigns those who responded too slowly to a single category of "no response." Distinct from all the other responses provided by the response latency, we must consider that the conditional probability assigned to this additional response category, given the latent trait, is non-zero. The conditional distribution of the item score is, therefore, partly discrete and partly continuous.

In the present paper, a general model in the closed response situation will be proposed and discussed. This model deals with the mixture of discrete and continuous item response distributions, and includes the general model for the open response situation as its special case. It also includes two more distinct situations, in which the response is half closed and half open. They will be introduced in the following section.

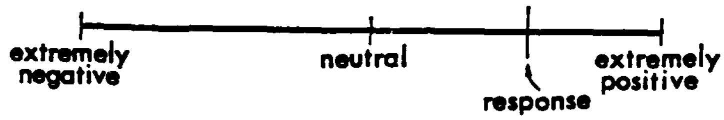

FIGURE 2-1

An Example of the Response Formats Which Allow Continuous Responses

II Closed/Open and Open/Closed Response Situations

Let θ be the unidimensional latent trait, or any hypothetical construct, which assumes all real numbers. Let g $(=1,2,...,n)$ be an item, which is the smallest, concrete entity devised for measuring the latent trait. The assumption that our latent space is unidimensional implies that the conditional or local independence of the item responses, given θ, holds in the unidimensional latent space.

In the previous paper (Samejima, 1973), the open and closed response situations are exemplified by a continuous response format, which is illustrated as Figure 2-1. Suppose that the subject is asked to check a point on a line segment illustrated in Figure 2-1, in accordance with his judgment required for the task in item g. We call it the open response situation if the subject is allowed to check any point of the line segment except for the two endpoints. Let z_g be the continuous item score, defined as the limiting case of the relative, graded item score, y_g, which is the raw item score x_g $(= 0, 1, ..., m_g)$ divided by the full item score m_g. Thus in the open response situation we have

$$(2.1) \qquad 0 < z_g < 1 ,$$

and z_g can be considered as the relative distance from the lower endpoint of the line segment illustrated in Figure 2-1. In the closed response situation, the subject is allowed to check any point of the line segment including the two endpoints. Thus we can write for this situation

$$(2.2) \qquad 0 \leq z_g \leq 1 .$$

In addition to the above two situations, we can conceive of two

more situations, which are half open and half closed. Suppose that the subject is allowed to check one of the endpoints of the response line illustrated by Figure 2-1, but not the other. We can write for these two situations

$$(2.3) \qquad 0 \leq z_g < 1$$

and

$$(2.4) \qquad 0 < z_g \leq 1 ,$$

respectively. Hereafter, we shall call the situation represented by (2.3) the closed/open response situation, and the one indicated by (2.4) the open/closed response situation. The illustration of the closed/open and open/closed response situations by means of Figure 2-1 is rather schematic than realistic. The reader may understand the implication of these two half-open and half-closed situations better, if we say that the example of response latency in the cognitive process, which was given in the preceding section, belongs to the closed/open response situation.

We notice that, if, regardless of the instructions, the nature of the item makes the subject refrain from checking either one of the two endpoints, or both, the general model for the open response situation will still be applicable in each of the other three situations. In addition, if the time limit is extremely long in the example of response latency given in the previous section, for instance, then the conditional probability assigned to "no response" will approach zero. In practice, however, it is more likely that we set a shorter time limit, so the conditional probability assigned to the lower endpoint becomes greater than zero. In fact, it is reported that in certain spatial problem solvings those who took much longer times on

difficult items than on easy items appeared to be relying on nonspatial skills (cf. Lohman and Kyllonen, 1983). Observations such as this support the necessity of time limits. For simplicity, in the following sections, we shall use the terms closed/open and open/closed response situations strictly for cases in which the probability assigned to one of the endpoints is non-zero, and the closed response situation indicates the case in which the probabilities assigned to the two endpoints are both non-zero.

Thus we must deal with the conditional distribution of the item score, given the latent trait θ, which is partly discrete and partly continuous. The distinction between the continuous response level and the discrete response level can no longer be made in terms of the continuous and discrete conditional distributions of the item score, given the latent trait. In this paper, we define the continuous response level as the one on which the conditional density of the item score, given the latent trait, is positive for the specified interval of the item score, except, at most, for an enumerable number of points. Thus the closed, the closed/open and the open/closed response situations belong to the continuous response level, as well as the open response situation, with positive infinity as the conditional density at $z_g = 0$ or $z_g = 1$, or both. It should also be noted that any continuous response can be transformed to z_g which varies between 0 and 1, regardless of the specified interval for which the original response is defined, provided that we choose a suitable monotone transformation.

III Conditional Distribution of the Item Score

Let $P^*_{z_g}(\theta)$ be the conditional probability with which the subject obtains the item score z_g or greater, given θ. A general mathematical

form for $P^*_{z_g}(\theta)$ in the homogeneous case of the continuous response model (Samejima, 1973) is given by

$$(3.1) \qquad P^*_{z_g}(\theta) = \int_{-\infty}^{a_g(\theta-b_{z_g})} \psi_g(t)\,dt \ ,$$

with

$$(3.2) \qquad \begin{cases} \lim_{\theta\to-\infty} P^*_{z_g}(\theta) = 0 \\ \lim_{\theta\to\infty} P^*_{z_g}(\theta) = 1 \end{cases} ,$$

where a_g (> 0) is the item discrimination parameter, b_{z_g} is the item response difficulty parameter, and $\psi_g(\cdot)$ is a specific continuous function, which characterizes the model, and is positive almost everywhere. The operating density characteristic, $H_{z_g}(\theta)$, has been defined, and it can be written in the form

$$(3.3) \qquad H_{z_g}(\theta) = a_g\ \psi_g\{a_g(\theta-b_{z_g})\}[\frac{d}{dz_g} b_{z_g}] \qquad 0 < z_g < 1 \ .$$

In the open response situation, this operating density characteristic is the density function of the conditional distribution of the item score z_g , given θ , which solely characterizes the conditional distribution of z_g , given θ , satisfying

$$(3.4) \qquad \int_0^1 H_{z_g}(\theta)\,dz_g = 1 \ .$$

Let $P_{z_g}(\theta)$ be the conditional probabilty of z_g , given θ . In the more general case which includes all the four response situations described earlier, we adopt (3.1) through (3.3) with the modification

presented by dotted curves are the corresponding five examples of (3.18), in which $\xi(z_g)$ is specified by (3.21) instead of (3.20), with the same set of parameter values. Since for $k = 1$ (3.21) coincides with (3.20), the dotted curve for $k = 1$ is overlapping with the solid curve which is based upon (3.20), and is not clearly visible in Figure 3-4.

Those are just some examples of the relationships between the continuous item score z_g and the difficulty parameter b_{z_g} in the closed/open response situation. In fact, we can conceive of many other strictly increasing functions varying from zero to unity than (3.20) and (3.21) for $\xi(z_g)$, and also many other strictly increasing functional formulae which vary from b_0 to positive infinity than (3.18) for the relationship between z_g and b_{z_g}. Figure 3-4 suggests, however, that even within the limitation of (3.18) with (3.20) or (3.21) for $\xi(z_g)$ we have varieties of curves which may fit our empirical findings well. In practice, we may use the arc tangent transformation of $(\hat{b}_{z_g} - \hat{b}_0)$, where $\hat{}$ indicates an estimate, and apply the method of moments for fitting polynomials (Samejima and Livingston, 1979) to the transformed values multiplied by $(2/\pi)$.

Figure 3-5 presents by a solid curve the operating density characteristic $H_{z_g}(\theta)$ in the normal ogive model as a function of the continuous item score z_g, for each of the thirteen fixed values of θ, i.e., -3.0, -2.5, -2.0, -1.5, -1.0, -0.5, 0.0, 0.5, 1.0, 1.5, 2.0, 2.5 and 3.5, with the parameters, $a_g = 1.0$ and $b_0 = -2.0$, using (3.18) as the functional relationship between z_g and b_{z_g} with the specification of $\xi(z_g)$ by (3.20) in which $k = 1$ and $\alpha_1 = 1.0$. In the same figure, also presented by dotted curves are the corresponding operating density characteristics in the logistic model, in which $D = 1.7$. Those graphs

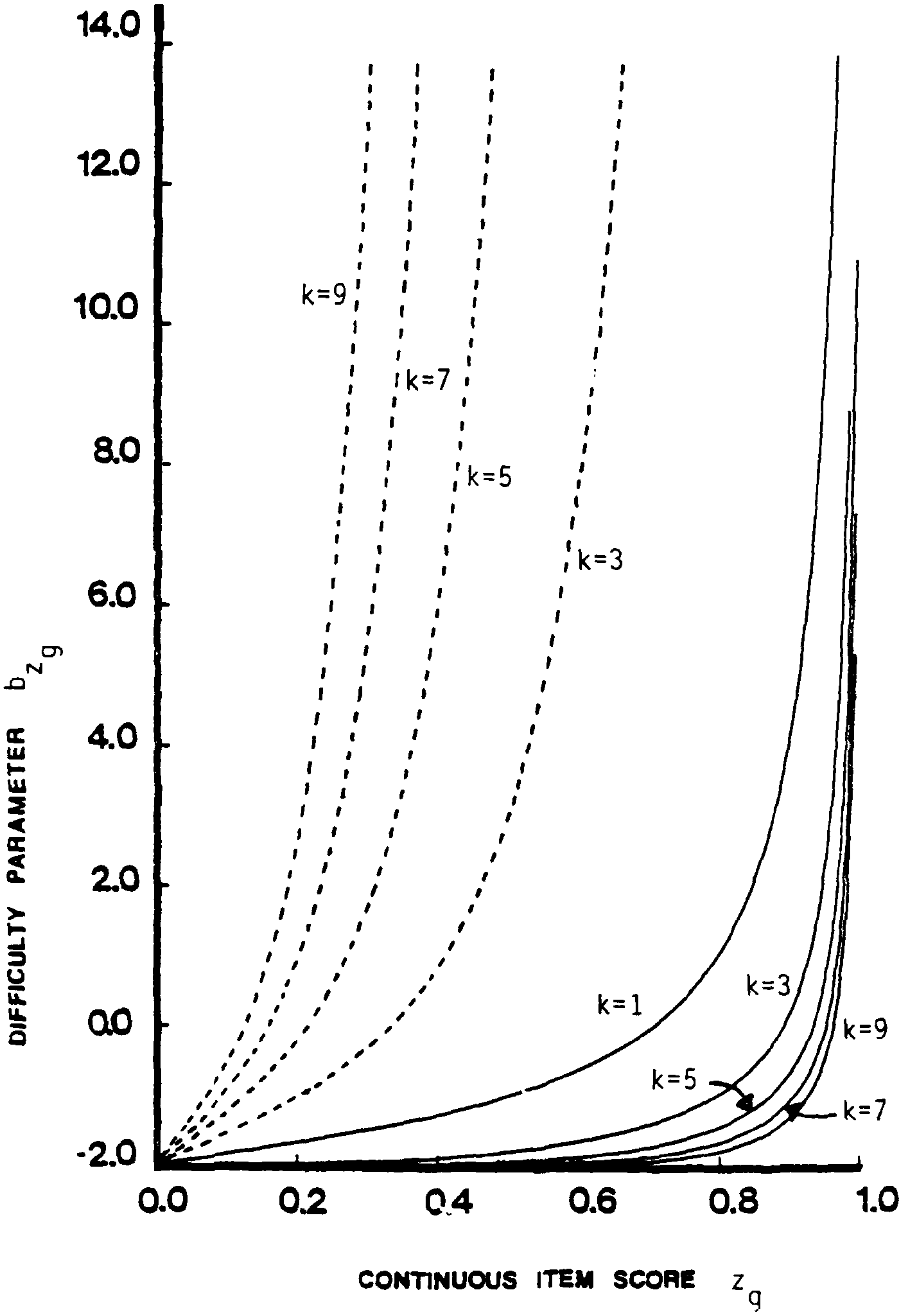

FIGURE 3-4

Five Hypothetical Functional Relationships (Solid Line) between the Continuous Item Score z_g and the Difficulty Parameter b_{z_g}, Which Are Given by $b_{z_g} = b_0 + \tan[(\pi/2)z_g^k]$ for $k = 1, 3, 5, 7, 9$, with the Parameter, $b_0 = -2.0$, and the Corresponding Relationships (Dotted Line) Given by $b_{z_g} = b_0 + \tan[(\pi/2)\{1-(1-z_g)^k\}]$. Closed/Open Response Situation.

closed/open response situation may be

$$(3.18) \qquad b_{z_g} = b_0 + \tan[(\pi/2)\ \xi(z_g)]$$

where $\xi(z_g)$ is any strictly increasing, continuous function of z_g defined for $0 \leq z_g \leq 1$, with the constraint

$$(3.19) \qquad \xi(z_g) \begin{cases} = 0 & z_g = 0 \\ = 1 & z_g = 1 \end{cases} .$$

Two examples of $\xi(z_g)$ are given by polynomials such that

$$(3.20) \qquad \xi(z_g) = \sum_{j=1}^{k} \alpha_j\, z_g^{\,j}$$

and

$$(3.21) \qquad \xi(z_g) = 1 - \sum_{j=1}^{k} \alpha_j (1-z_g)^j \quad ,$$

with the constraints given by the right hand inequality of (3.16) and

$$(3.22) \qquad \sum_{j=1}^{k} \alpha_j = 1 \quad .$$

Figure 3-4 presents by solid curves five examples of the above functional relationship with (3.20) as $\xi(z_g)$ and with $b_0 = -2.0$, where $k = 1, 3, 5, 7, 9$ and $\alpha_j = 0$ for $j < k$. In this figure, it is obvious that b_{z_g} approaches positive infinity as z_g tends to unity, and it approaches b_0 as z_g tends to zero. In the same figure, also

with the same constraints given by (3.15) and (3.16). Five examples of (3.17) with the same values of k and the coefficients α_j are drawn by dotted lines in Figure 3-3. Note that, in the first of the five examples, (3.17) coincides with the linear function given by (3.11), and the dotted line is overlapping with the solid line in Figure 3-3.

We can see from (3.3) that, as far as $\Psi_g(\cdot)$ is symmetric and $b_0 = -b_1$, as is the case with the present examples, the operating density characteristic, $H_{z_g}(\theta)$, which is based upon (3.14) for the relationship between z_g and b_{z_g}, assumes the same value as the corresponding operating density characteristic based upon (3.17) with the same values of coefficients, if we replace z_g by $(1-z_g)$ and θ by $-\theta$. This implies that the mirror images of the thirteen curves for $H_{z_g}(\theta)$ with the axis of rotation at $z_g = 0.5$ in each of the four figures, Figures A-1-2 through A-1-5, provide us with the operating density characteristics based upon each of the four dotted curves in Figure 3-3 for the relationship between z_g and b_{z_g}, for $\theta = -3.5, -2.5, -2.0, -1.5, -1.0, -0.5, 0.0, 0.5, 1.0, 1.5, 2.0, 2.5, 3.0$, in each of the two models.

In contrast to the observations made so far in the closed response situation, neither in the closed/open response situation nor in the open/closed response situation can the functional relationship between the item score z_g and the difficulty parameter b_{z_g} be linear, nor can it be approximated by a polynomial. This is obvious from (3.6) and from the fact that, in the former situation, inequality holds only in the first formula of (3.6), and it holds only in the second formula in the latter situation. We must search in practice, therefore, for some other functional formulae in each situation, after having observed the empirical function obtained between z_g and the estimated b_{z_g} from our data. One suitable formula in the

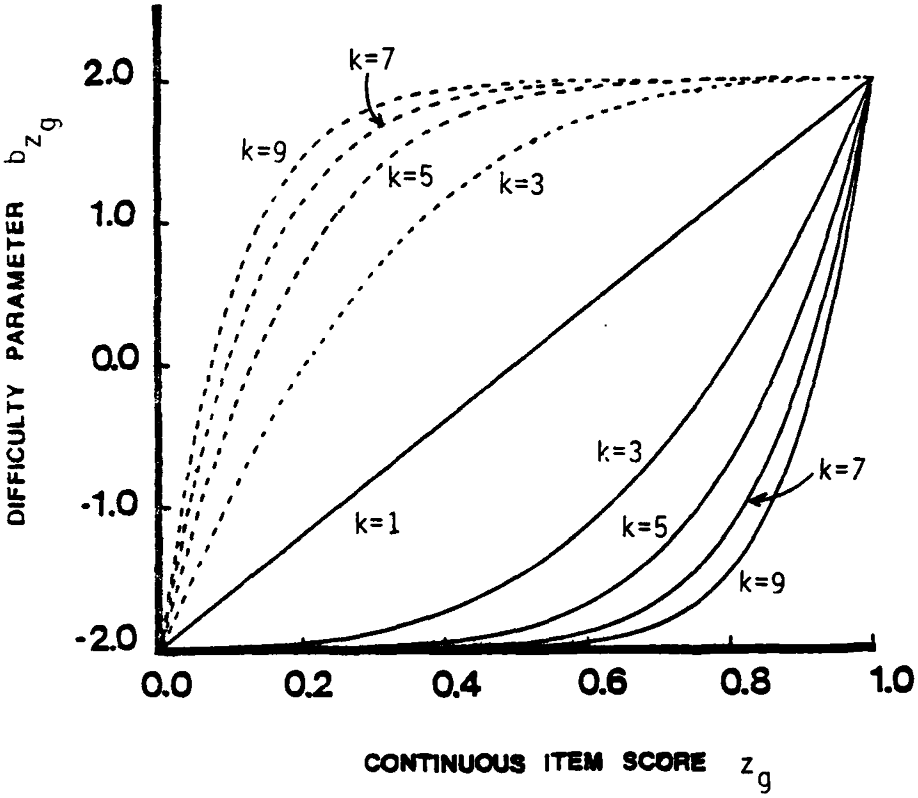

FIGURE 3-3

Five Hypothetical Functional Relationships (Solid Line) between the Continuous Item Score z_g and the Difficulty Parameter b_{z_g}, Which Are Given by $b_{z_g} = b_0+(b_1-b_0)z_g^k$, for $k = 1, 3, 5, 7, 9$, with the Parameters, $b_0 = -2.0$ and $b_1 = 2.0$, and the Corresponding Relationships (Dotted Line) Given by $b_{z_g} = b_1-(b_1-b_0)(1-z_g)^k$. Closed Response Situation.

necessary, condition for the second constraint to hold is that $\alpha_j \geq 0$ for $j = 1, 2, \ldots, k$. We notice that the linear relationship which we observed earlier is included by (3.14) with the specification, $k = 1$. Figure 3-3 illustrates the functional relationships between z_g and b_{z_g} for five different polynomials in which $k = 1, 3, 5, 7, 9$ and $\alpha_j = 0$ for $j < k$, with $b_0 = -2.0$ and $b_1 = 2.0$. It is noted that the resultant $H_{z_g}(\theta)$ based upon the linear relationship shown in Figure 3-3 has been given in Figure 3-2 for the thirteen different values of θ, in both the normal ogive and the logistic models. The corresponding sets of $H_{z_g}(\theta)$ for the other four relationships between z_g and b_{z_g}, which are shown in Figure 3-3, are presented in Appendix I as Figures A-1-2 through A-1-5. In those figures, for convenience, the continuous item score z_g is taken on the ordinate, and the operating density characteristic $H_{z_g}(\theta)$ is taken on the abscissa, and, because of the limitation of the space, some curves are truncated. For the sake of comparison, Figure 3-2 is duplicated as Figure A-1-1 in Appendix I with the exchange of the ordinate and the abscissa. It is noted that the middle term of (3.5) is not influenced by the specific functional relationship between z_g and b_{z_g}, so it is obvious that the thirteen areas shown in Table 3-1 also represent those under the thirteen curves shown in each of the four sets of graphs, Figures A-1-2 through A-1-5, in each of the two models, respectively.

We notice that, if we use a set of coefficients satisfying $\alpha_j \geq 0$ for $j = 1, 2, \ldots, k$, those specific polynomials defined by (3.14) with the constraints, (3.15) and (3.16), are always convex. A set of concave polynomials can be obtained under the same condition by

$$(3.17) \qquad b_{z_g} = b_1 - \sum_{j=1}^{k} \alpha_j (1-z_g)^j$$

TABLE 3-1

Areas Under the Operating Density Characteristic, $H_{z_g}(\theta)$, for the Interval, $0 < z_g < 1$, in Each of the Normal Ogive (N. O.) and the Logistic (LOG.) Models, and in Each of the Closed, the Closed/Open and the Open/Closed Response Situations with the Discrimination Parameter $a_g = 1.0$, and the Difficulty Parameter $b_0 = -2.0$ or $b_1 = 2.0$, or Both, for Thirteen Different Fixed Values of θ. For the Scaling Factor in the Logistic Model, $D = 1.7$ Is Used.

θ	Closed		Closed/Open		Open/Closed	
	N.O.	LOG.	N.O.	LOG.	N.O.	LOG.
-3.0	0.15866	0.15426	0.15866	0.15447	1.00000	0.99980
-2.5	0.30853	0.29896	0.30854	0.29943	1.00000	0.99952
-2.0	0.49997	0.49889	0.50000	0.50000	0.99997	0.99889
-1.5	0.69123	0.69797	0.69146	0.70057	0.99977	0.99740
-1.0	0.83999	0.83947	0.84134	0.84553	0.99865	0.99394
-0.5	0.92698	0.91351	0.93319	0.92757	0.99379	0.98594
0.0	0.95450	0.93541	0.97725	0.96770	0.97725	0.96770
0.5	0.92698	0.91351	0.99379	0.98594	0.93319	0.92757
1.0	0.83999	0.83947	0.99865	0.99394	0.84134	0.84553
1.5	0.69123	0.69797	0.99977	0.99740	0.69146	0.70057
2.0	0.49997	0.49889	0.99997	0.99889	0.50000	0.50000
2.5	0.30853	0.29896	1.00000	0.99952	0.30854	0.29943
3.5	0.06681	0.07234	1.00000	0.99991	0.06681	0.07243

$P_0(\theta)$ and $P_1(\theta)$ for the specified latent trait θ. From (3.1), (3.7) and (3.8) we can see that this area assumes a maximal value at $\theta = (b_0+b_1)/2$, provided that $\Psi_g(\cdot)$ is a symmetric and unimodal function, as is the case with both the normal ogive and the logistic models which are characterized by (3.9) and (3.10), respectively. Since we used $b_0 = -2.0$ and $b_1 = 2.0$ in our examples shown in Figure 3-2, it is obvious that this area is at the maximum for $\theta = 0.0$. Those areas for our thirteen examples are shown in Table 3-1, in each of the two models.

The relationship between z_g and b_{z_g} can be any strictly increasing function other than the linear function, with the constraint that b_{z_g}'s are a priori specified at $z_g = 0$ and $z_g = 1$. For practical purposes, it may be appropriate to consider various strictly increasing polynomials for approximations to such functional relationships. We can write for such polynomials of degree k

$$(3.14) \qquad b_{z_g} = b_0 + \sum_{j=1}^{k} \alpha_j z_g^{\,j}$$

with the two constraints,

$$(3.15) \qquad \sum_{j=1}^{k} \alpha_j = b_1 - b_0$$

and

$$(3.16) \qquad \frac{d}{dz_g} b_{z_g} = \sum_{j=1}^{k} \alpha_j \, j \, z_g^{\,j-1} \geq 0 \qquad\qquad 0 < z_g < 1 ,$$

where strict inequality holds for all values of z_g between zero and unity, except, at most, at an enumerable number of points. A sufficient, though not

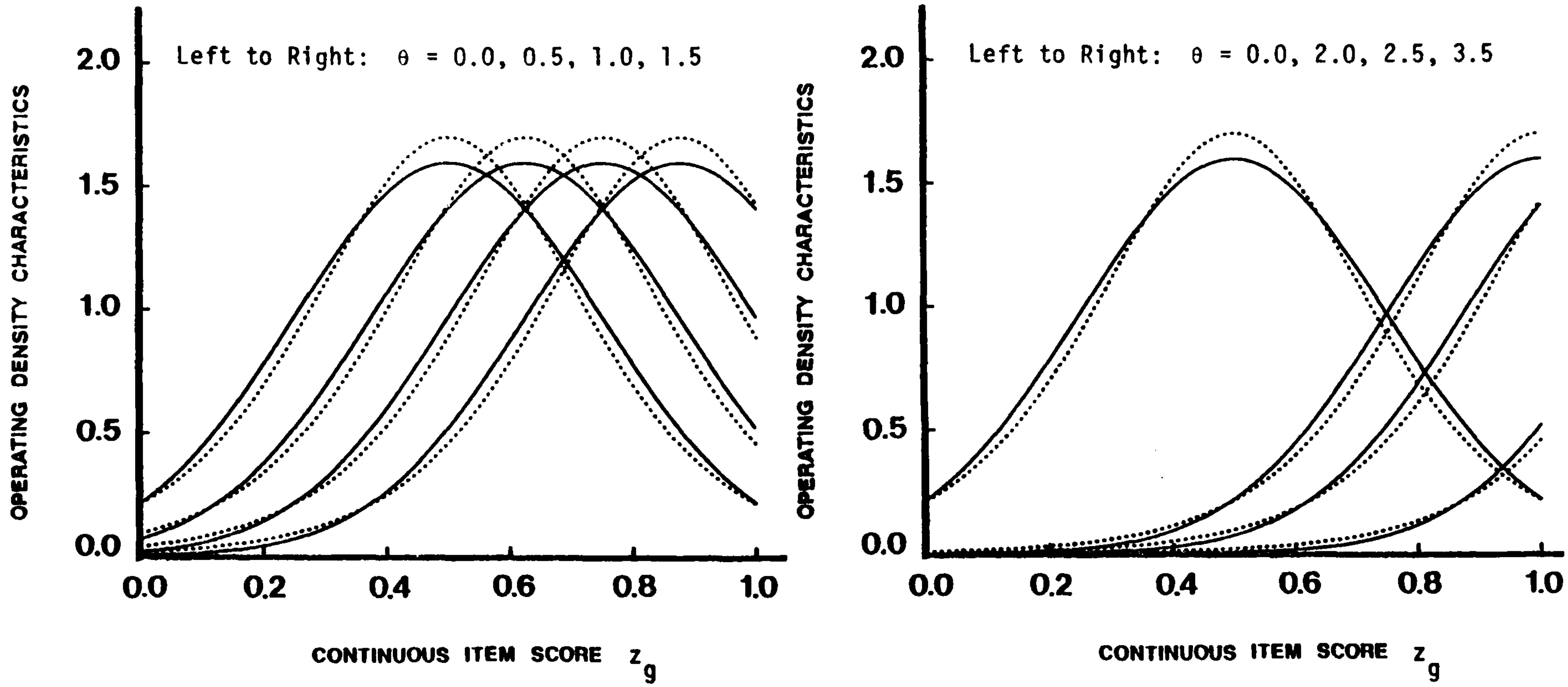

FIGURE 3-2 (Continued)

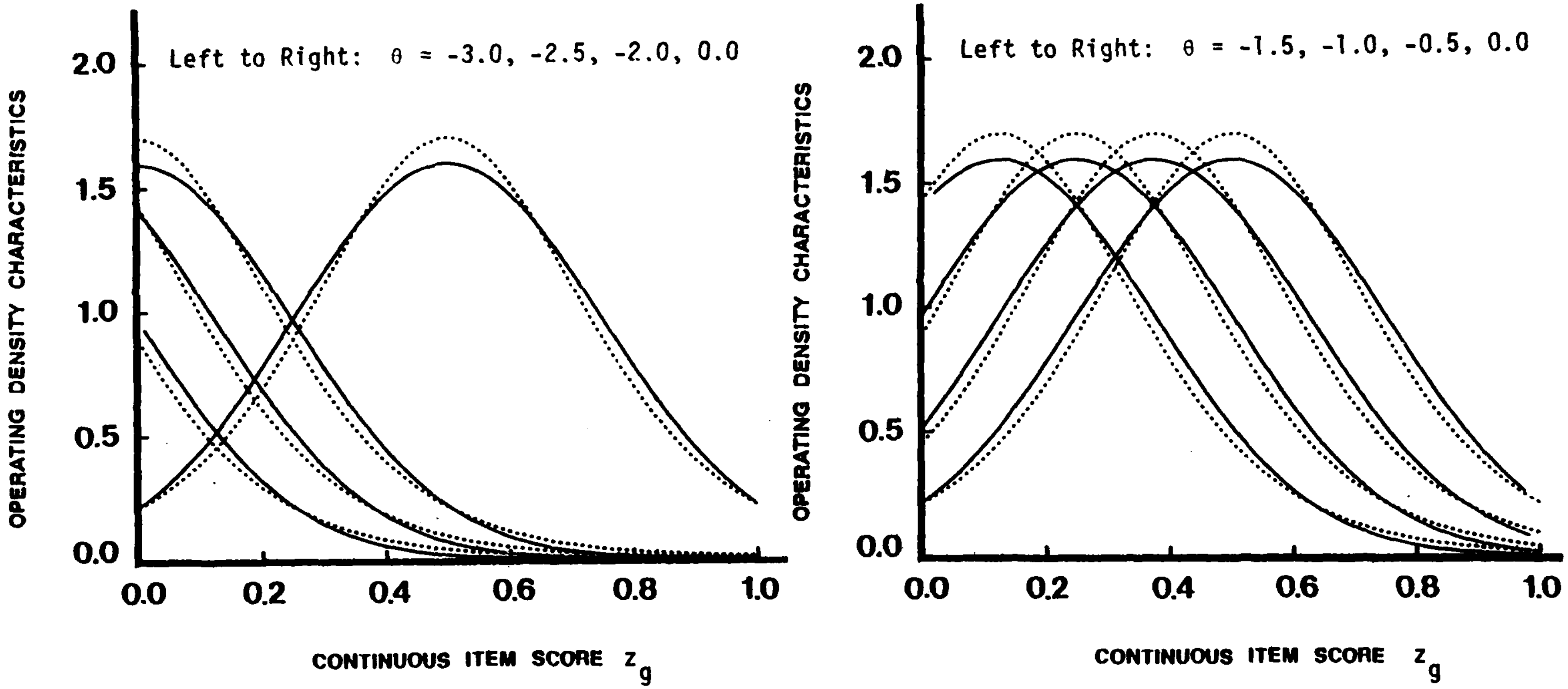

FIGURE 3-2

Operating Density Characteristic, $H_{z_g}(\theta)$, for Each of the Thirteen Values of θ , i.e., -3.0, -2.5, -2.0, -1.5, -1.0, -0.5, 0.0, 0.5, 1.0, 1.5, 2.0, 2.5 and 3.5 , Following the Normal Ogive Model with $a_g = 1.0$, $b_0 = -2.0$ and $b_1 = 2.0$, with the Linear Relationship Between the Item Score z_g and the Difficulty Parameter b_{z_g} , Represented by a Solid Curve. Corresponding Thirteen Functions Following the Logistic Model with $D = 1.7$ are also Drawn by Dotted Curves.
Closed Response Situation.

Substituting (3.12) into (3.3) and rearranging, we have for the operating density characteristic of z_g

$$H_{z_g}(\theta) = a_g(b_1-b_0)\ \psi_g\{a_g[\theta-b_0-(b_1-b_0)z_g]\} \ . \tag{3.13}$$

It is obvious from (3.13) that the operating density characteristics, $H_{z_g}(\theta)$, when considered as functions of z_g , are identical for different fixed values of θ , except for the positions on the axis z_g . In addition, if $\psi_g(\cdot)$ is symmetric and unimodal, so is $H_{z_g}(\theta)$, although it may be truncated, and the modal point is $(\theta-b_0)/(b_1-b_0)$, provided that θ is greater than, or equal to, b_0 . Figure 3-2 illustrates by solid lines thirteen curves for $H_{z_g}(\theta)$ in the normal ogive model with the parameters, $a_g = 1.0$, $b_0 = -2.0$ and $b_1 = 2.0$, in the closed response situation for thirteen different values of θ , i.e., -3.0 , -2.5 , -2.0 , -1.5 , -1.0 , -0.5 , 0.0 , 0.5 , 1.0 , 1.5 , 2.0 , 2.5 and 3.5 , when the linear relationship between z_g and b_{z_g} holds. In the same figure, also presented by dotted lines are the corresponding thirteen curves for $H_{z_g}(\theta)$ following the logistic model, with the same parameter values and with 1.7 substituting for D in (3.10). It is well known (Birnbaum, 1968) that, if we use this value for the scaling factor D , then $P^*_{z_g}(\theta)$ in the logistic model will become very close to the corresponding $P^*_{z_g}(\theta)$ in the normal ogive model. We can see in Figure 3-2 that each of the thirteen curves for $H_{z_g}(\theta)$ in the logistic model is close to the corresponding curve in the normal ogive model, although the former is a little steeper.

It is obvious from (3.5) that the area under the curve of the operating density characteristic, $H_{z_g}(\theta)$, depends upon the values of

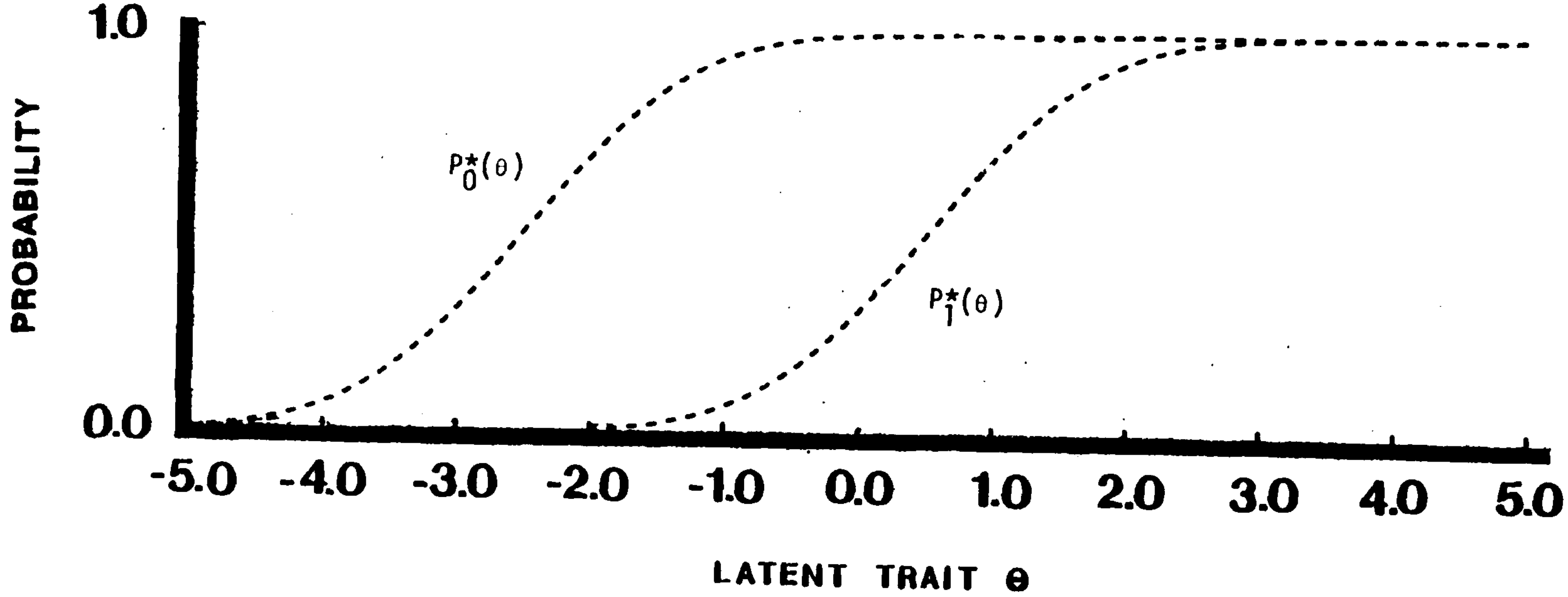

FIGURE 3-1

$P_{z_g}^*(\theta)$, or the Conditional Probability with Which the Subject Obtains the Item Score z_g , in the Closed Response Situation for $z_g = 0$ and $z_g = 1$. In This Example, the Normal Ogive Model is Used, with $a_g = 1.0$, $b_0 = -2.6$ and $b_1 = 0.5$.

Figure 3-1 illustrates $P^*_0(\theta)$ and $P^*_1(\theta)$ when inequality holds in each of the two formulae of (3.6), i.e., in the closed response situation. Note that in Figure 3-1 the two curves for $P^*_0(\theta)$ and $P^*_1(\theta)$ are identical, except for the position on the abscissa, as is the characteristic of the homogeneous case (Samejima, 1972, 1973). In this example, the two curves follow the <u>normal ogive model</u>, which is characterized by

$$\Psi_g(t) = (2\pi)^{-1/2}\exp(-t^2/2) \ , \tag{3.9}$$

with the parameters $a_g = 1.0$, $b_0 = -2.6$ and $b_1 = 0.5$. If $\Psi_g(t)$ is defined by

$$\Psi_g(t) = D\exp(-Dt)\ [1+\exp(-Dt)]^{-2} \ , \tag{3.10}$$

where D is a scaling factor, we shall call the specified model the <u>logistic model</u>.

It is obvious from (3.3) that the operating density characteristic, $H_{z_g}(\theta)$, depends heavily upon the relationship between the item score z_g and the difficulty parameter b_{z_g} , as well as on the functional formula $\Psi_g(\cdot)$. For the purpose of illustration, let us consider the case in which b_{z_g} is a linear function of the item score z_g . We can write

$$b_{z_g} = b_0 + (b_1-b_0)z_g \ , \tag{3.11}$$

and from this we obtain

$$\frac{d}{dz_g} b_{z_g} = b_1 - b_0 \ . \tag{3.12}$$

of (3.4) into the form

$$\int_0^1 H_{z_g}(\theta)\, dz_g = 1 - [P_0(\theta) + P_1(\theta)] \leqslant 1 \quad , \tag{3.5}$$

where $P_0(\theta)$ and $P_1(\theta)$ indicate $P_{z_g}(\theta)$ for $z_g = 0$ and $z_g = 1$, respectively. We can also write for the difficulty parameter b_{z_g}

$$\begin{cases} \lim\limits_{z_g \to 0} b_{z_g} = b_0 \geqslant -\infty \\ \lim\limits_{z_g \to 1} b_{z_g} = b_1 \leqslant \infty \end{cases} \quad . \tag{3.6}$$

We obtain from the definitions of $P_{z_g}(\theta)$ and $P^*_{z_g}(\theta)$

$$P_{z_g}(\theta) \begin{cases} = Q^*_{z_g}(\theta) & z_g = 0 \\ = P^*_{z_g}(\theta) & z_g = 1 \end{cases} , \tag{3.7}$$

where

$$Q^*_{z_g}(\theta) = 1 - P^*_{z_g}(\theta) . \tag{3.8}$$

It is noted that in the open response situation, $P_0(\theta) = 0$ and $P_1(\theta) = 0$ throughout the whole range of θ , and an equality holds in (3.5) and in each formula of (3.6). In each of the other three situations, i.e., the closed response situation, the closed/open response situation and the open/closed response situation, however, equality does not hold in one of the formulae of (3.6), or in either of them, and the left hand side of (3.5) becomes less than unity. In such cases the conditional distribution of the item score z_g , given θ , is discrete at $z_g = 0$ or at $z_g = 1$, or both, and continuous for $0 < z_g < 1$.

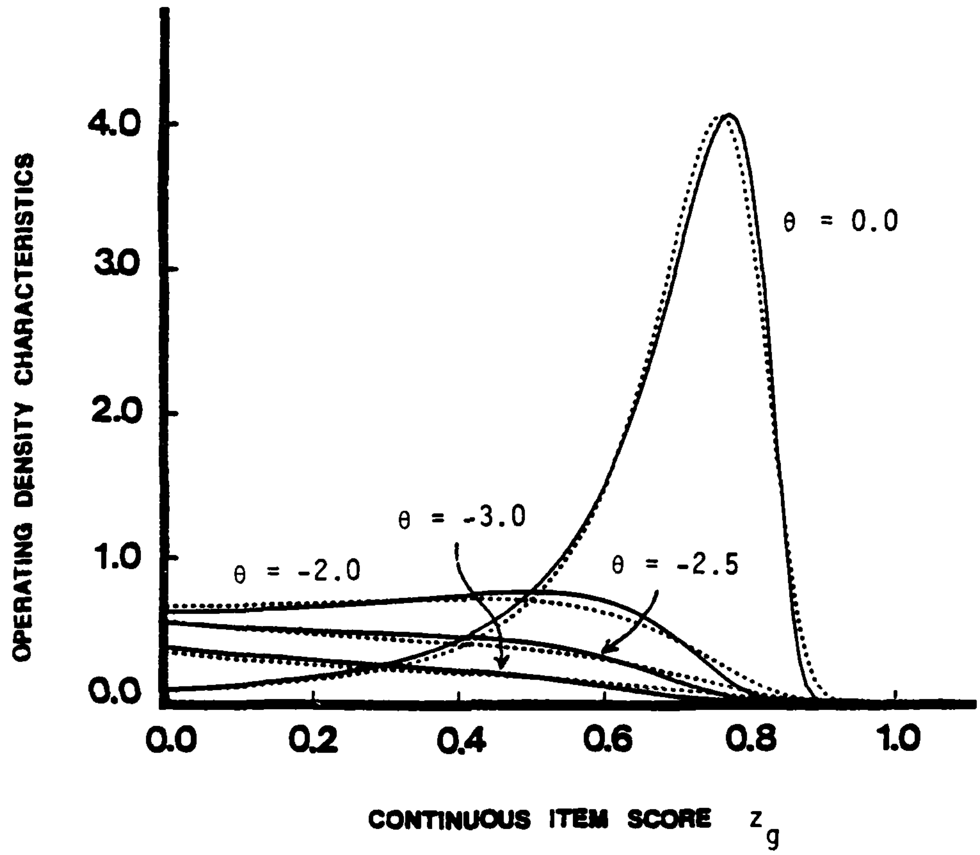

FIGURE 3-5

Operating Density Characteristic, $H_{z_g}(\theta)$, for Each of the Thirteen Values of θ, i.e., -3.0, -2.5, -2.0, -1.5, -1.0, -0.5, 0.0, 0.5, 1.0, 1.5, 2.0, 2.5, and 3.5, Following the Normal Ogive Model with $a_g = 1.0$ and $b_0 = -2.0$, with $b_{z_g} = b_0 + \tan[(\pi/2)z_g]$, Represented by a Solid Curve. Corresponding Thirteen Functions Following the Logistic Model with $D = 1.7$ are also Drawn by Dotted Curves. Closed/Open Response Situation.

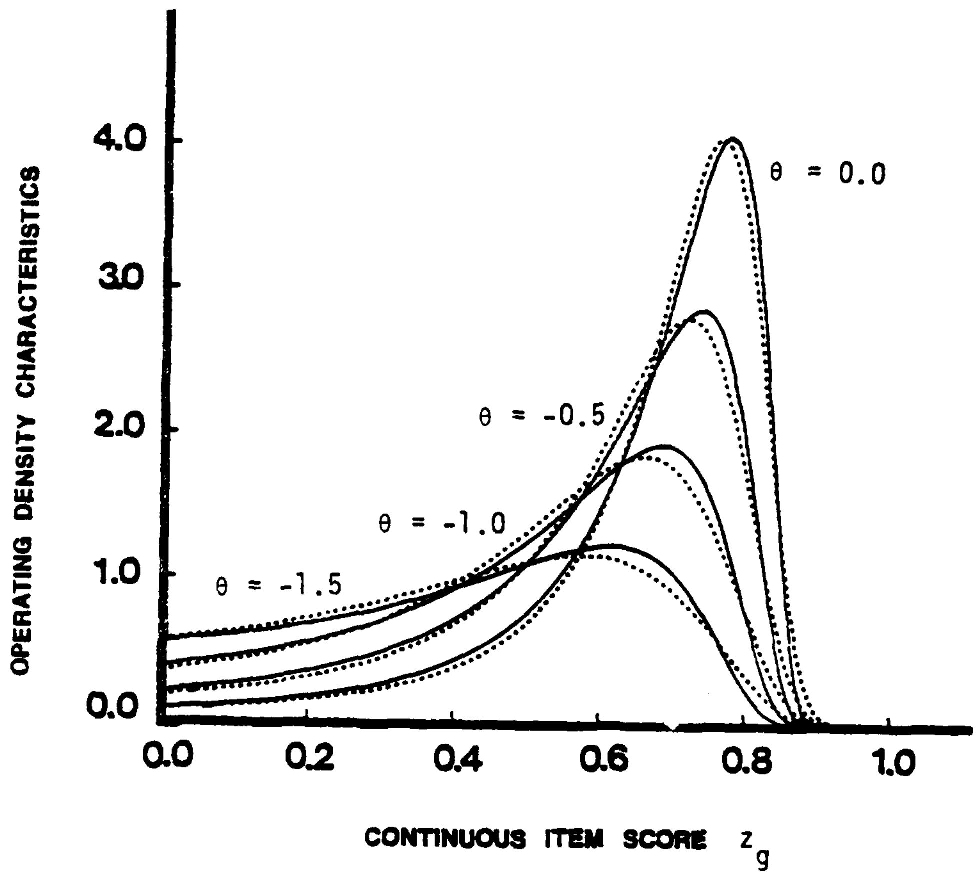

FIGURE 3-5 (Continued)

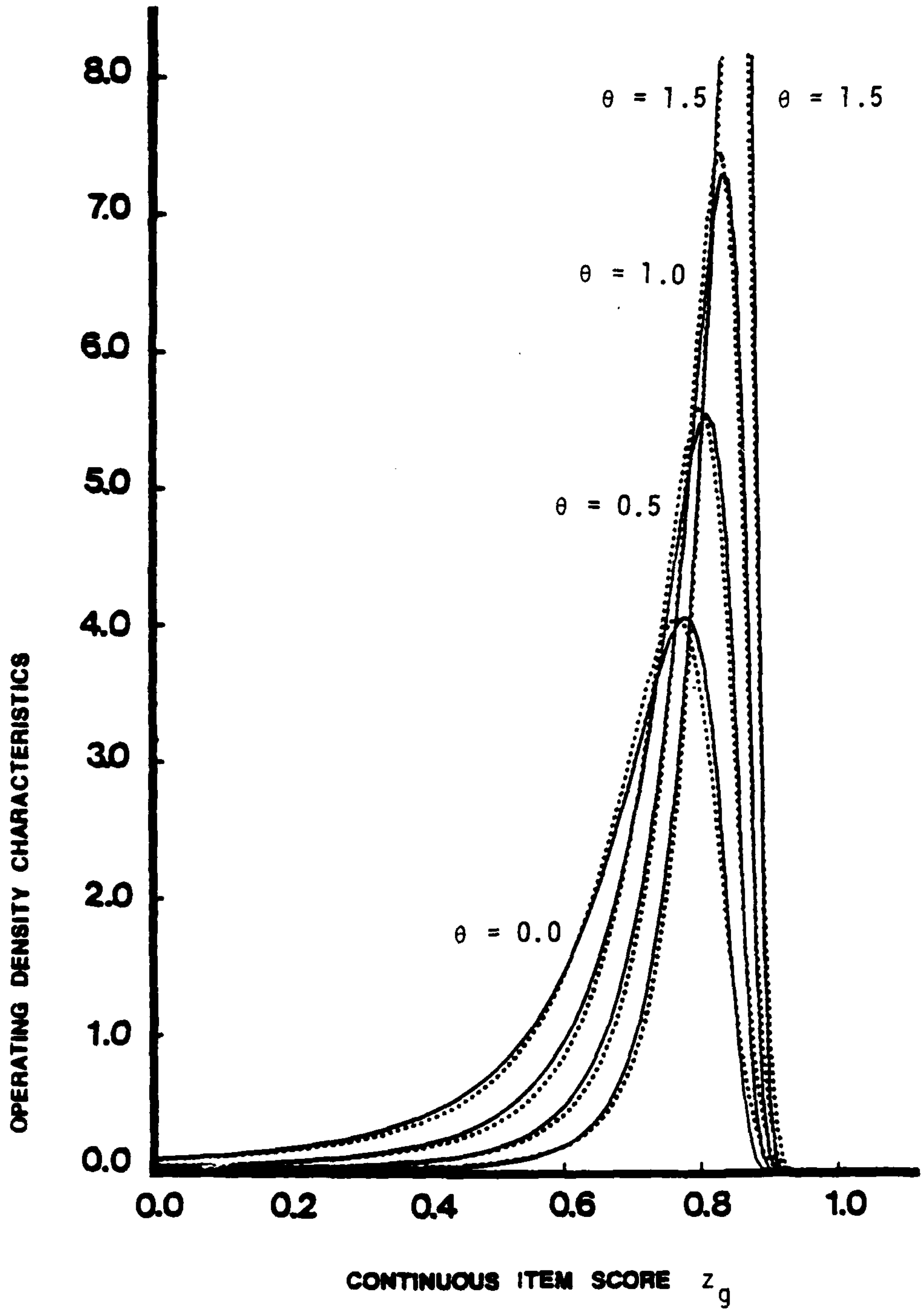

FIGURE 3-5 (Continued)

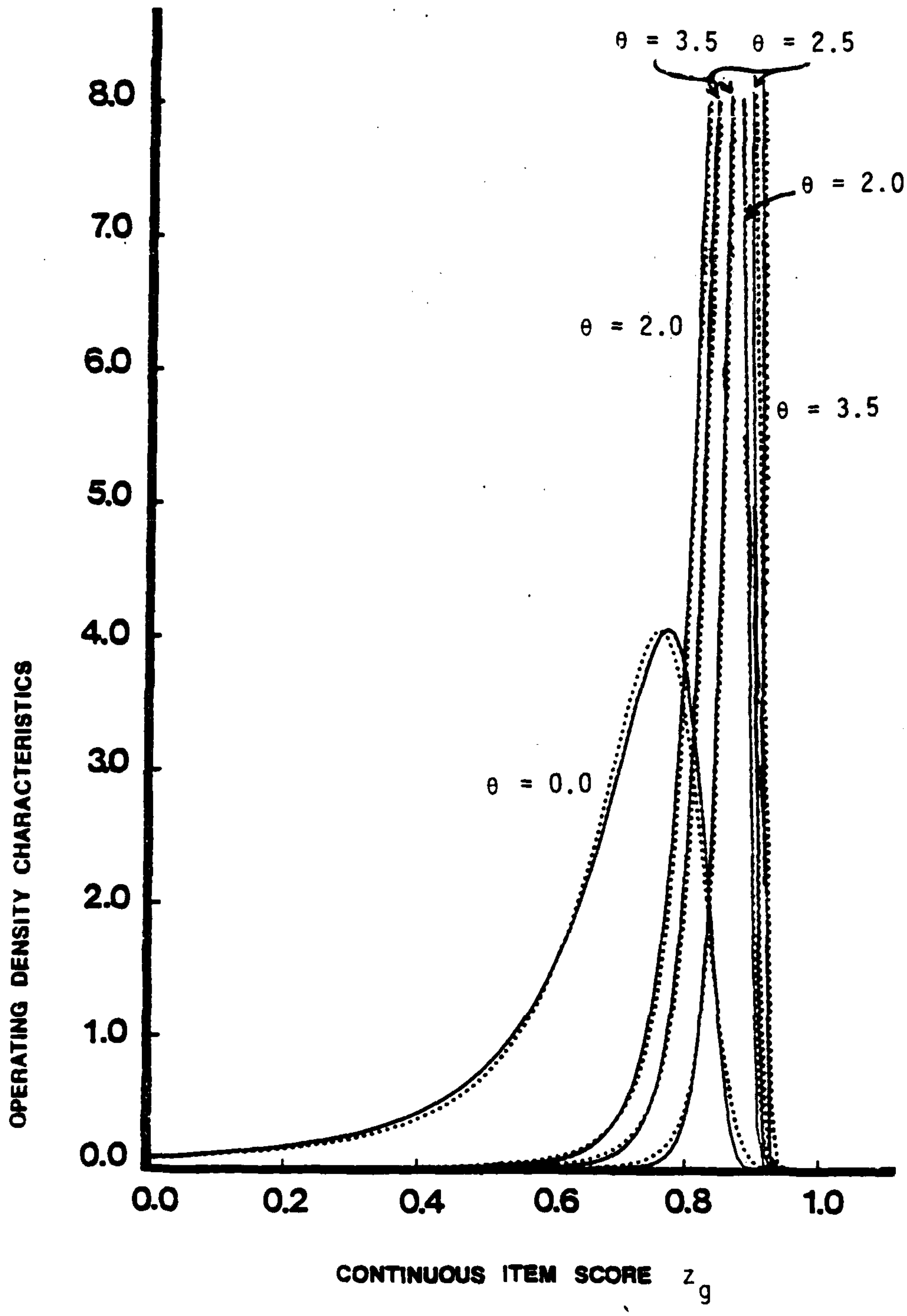

FIGURE 3-5 (Continued)

and four more sets of graphs with $\alpha_k = 1.0$ for $k = 3, 5, 7, 9$, respectively, are given in Appendix II as Figures A-2-1 through A-2-5, with the reversal of the abscissa and the ordinate.

It is obvious from (3.5) that, in the closed/open response situation, the area under the operating density characteristic depends solely upon the value of $P_0(\theta)$, and is independent of the particular functional formula for the relationship between the continuous item score z_g and the difficulty parameter b_{z_g}. Thus in each model the five areas shown in Figures A-2-1 through A-2-5 for each specific value of θ are equal. Those areas for the thirteen different values of θ are also shown in Table 3-1, in each of the normal ogive and the logisitc models.

Similarly, in the open/closed response situation, one useful formula for the relationship between the continuous item score z_g and the difficulty parameter b_{z_g} may be

$$(3.23) \qquad b_{z_g} = b_1 + \tan[(-\pi/2)\ \zeta(z_g)] \quad ,$$

where $\zeta(z_g)$ is any strictly decreasing, continuous function of z_g defined for $0 \leq z_g \leq 1$, with the constraint

$$(3.24) \qquad \zeta(z_g) \begin{cases} = 1 & z_g = 0 \\ = 0 & z_g = 1 \end{cases} \quad .$$

Again, for practical purposes, it may suffice if we consider polynomials such that

$$(3.25) \qquad \zeta(z_g) = \sum_{j=1}^{k} \alpha_j (1-z_g)^j$$

or

$$(3.26) \qquad \zeta(z_g) = 1 - \sum_{j=1}^{k} \alpha_j z_g^{\,j} \ ,$$

where k is the degree of polynomial and α_j $(j=1,2,\ldots,k)$ is a coefficient, with the constraints given by the right hand inequality of (3.16) and (3.22) .

Figure 3-6 presents by solid curves the five examples of the functional relationships given by (3.23), with (3.25) for $\zeta(z_g)$, in which $b_1 = 2.0$ and $\alpha_k = 1.0$ for $k = 1, 3, 5, 7, 9$, respectively. As we can see in this figure, they are strictly increasing functions of z_g with negative infinity and b_1 as the two asymptotes. In the same figure, also presented by dotted curves are the corresponding five difficulty parameter functions based upon (3.26) instead of (3.25), specified by the same set of parameter values. We notice that Figure 3-6 can be obtained by rotating Figure 3-4 about the point (0.5, 0.0) by the angle π , the fact that is implied in (3.18) and (3.23). Again for $k = 1$, (3.26) coincides with (3.25), so the dotted curve for $k = 1$ is overlapping with the corresponding solid curve based upon (3.25) in Figure 3-6. As we have seen in the closed/open response situation, we find in Figure 3-6 that even within the limitation of the functional formula given by (3.23) with (3.25) or (3.26) for $\zeta(z_g)$ we shall have varieties of curves which may fit our empirical findings well, although it is just one example of many conceivable functional relationships between the continuous item score z_g and the difficulty paramenter b_{z_g} in the open/closed response situation.

Figure 3-7 presents the operating density characteristic $H_{z_g}(\theta)$ in the normal ogive model and that in the logistic model as functions of the continuous item score z_g by solid and dotted curves, respectively, for

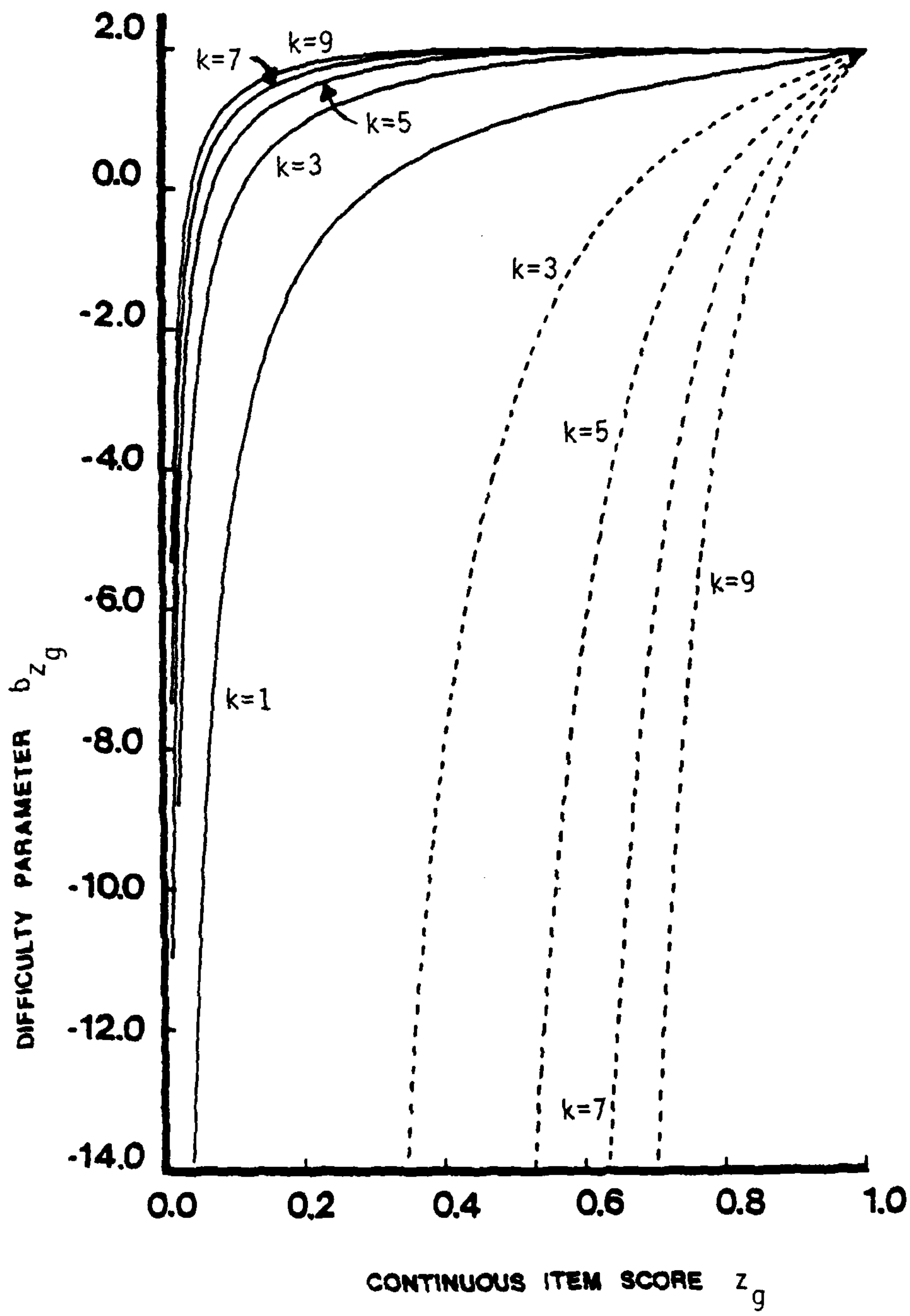

FIGURE 3-6

Five Hypothetical Functional Relationships (Solid Line) between the Continuous Item Score z_g and the Difficulty Parameter b_{z_g}, Which Are Given by $b_{z_g} = b_1 + \tan[(-\pi/2)(1-z_g)^k]$ for $k = 1, 3, 5, 7, 9$, with the Parameter, $b_1 = 2.0$, and the Corresponding Relationships (Dotted Line) Given by $b_{z_g} = b_1 + \tan[(-\pi/2)(1-z_g^k)]$. Open/Closed Response Situation.

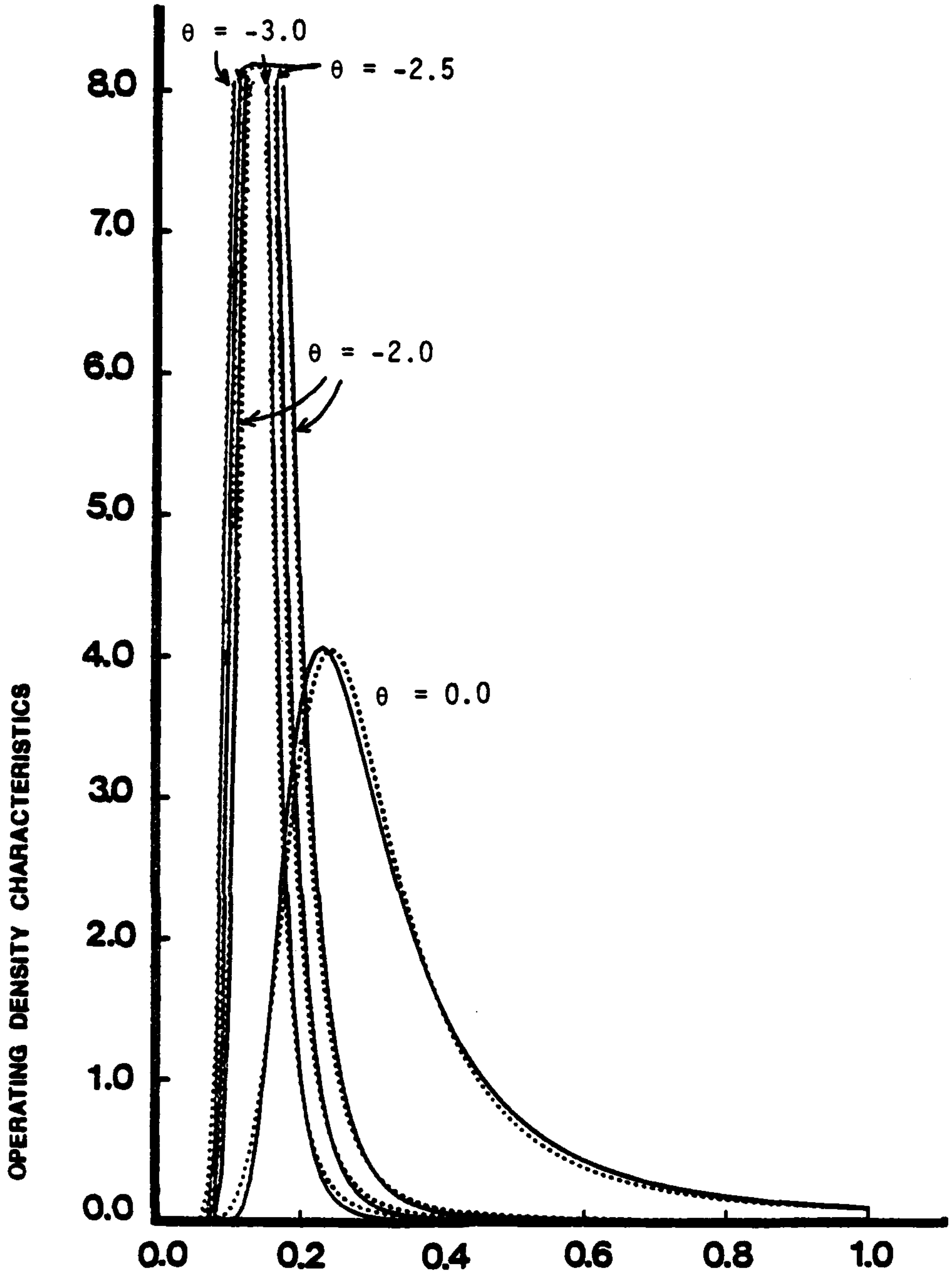

FIGURE 3-7

Operating Density Characteristic, $H_{z_g}(\theta)$, for Each of the Thirteen Values of θ, i.e., -3.0 , -2.5 , -2.0 , -1.5 , -1.0 , -0.5 , 0.0 , 0.5 , 1.0 , 1.5 , 2.0 , 2.5 , and 3.5 , Following the Normal Ogive Model with $a_g = 1.0$ and $b_1 = 2.0$, with $b_{z_g} = b_1 + \tan[-(\pi/2)(1-z_g)]$, Represented by a Solid Curve. Corresponding Thirteen Functions Following the Logistic Model with $D = 1.7$ are also Drawn by Dotted Curves. Open/Closed Response Situation.

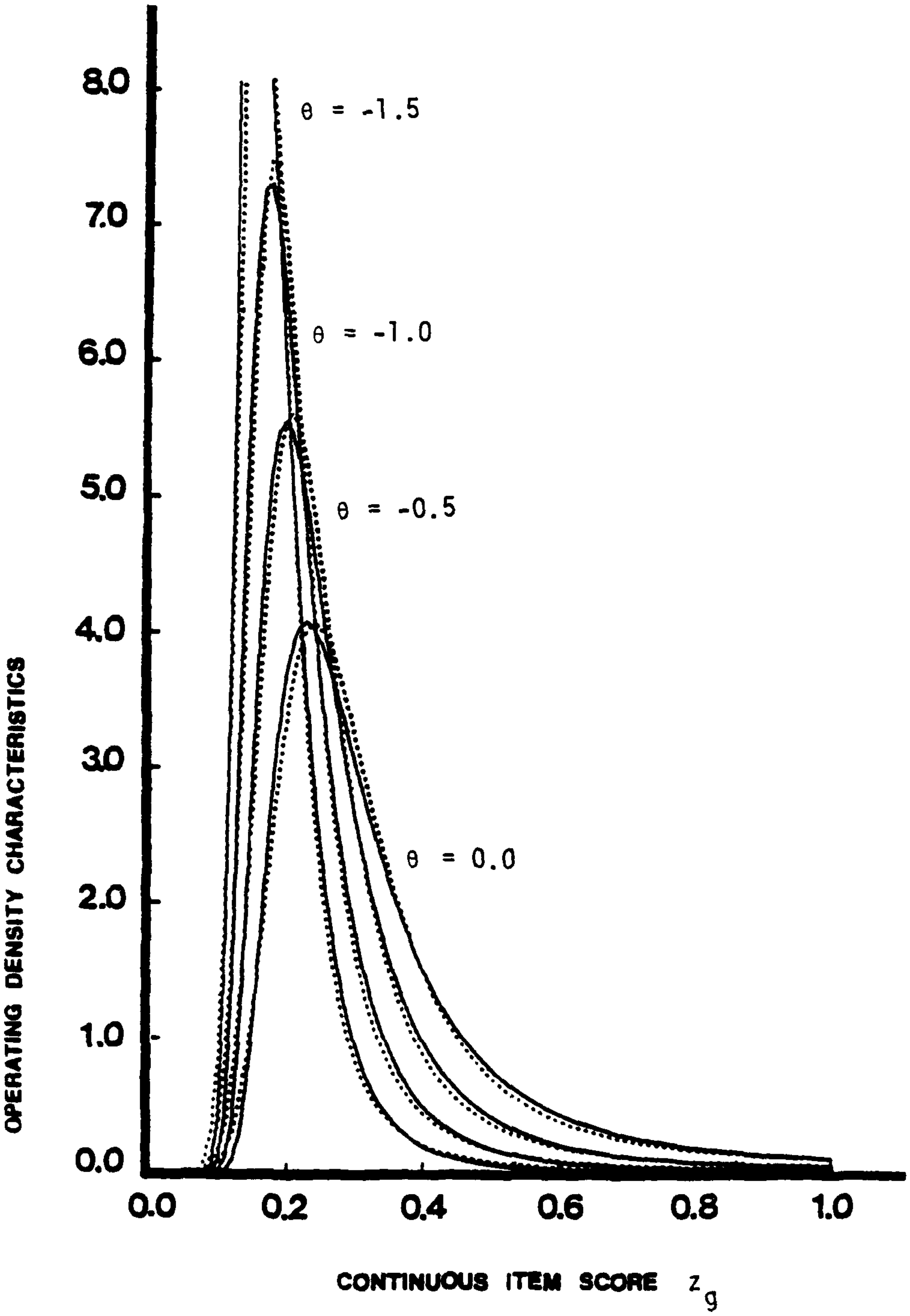

FIGURE 3-7 (Continued)

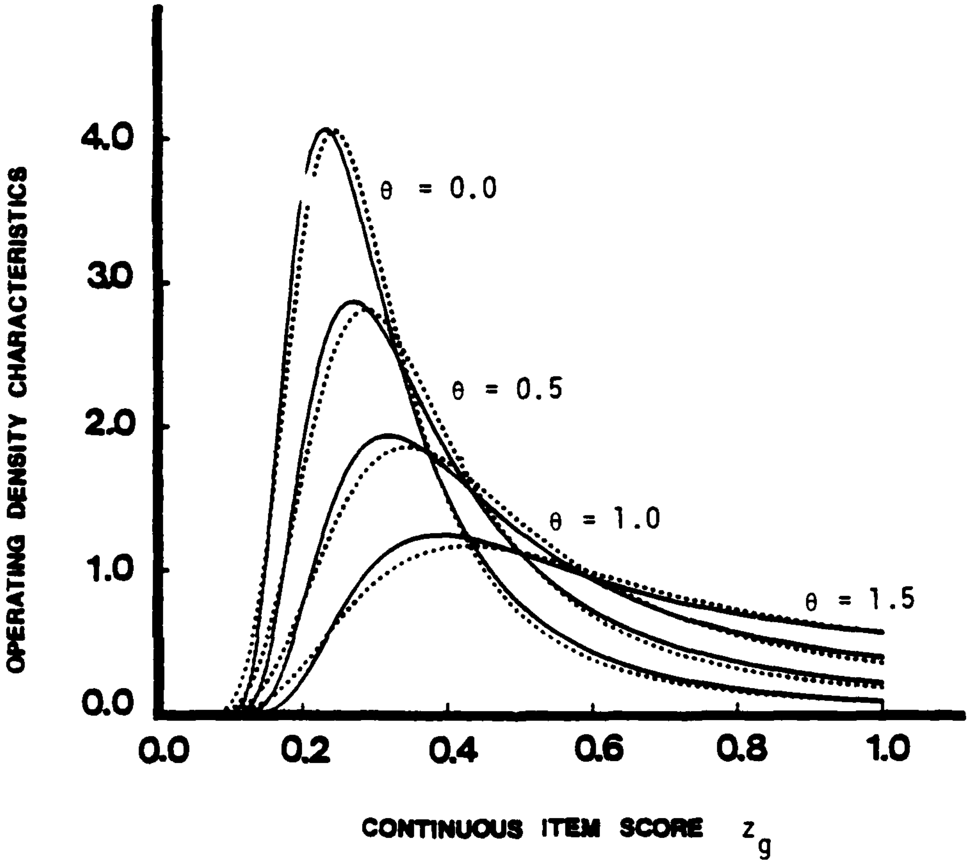

FIGURE 3-7 (Continued)

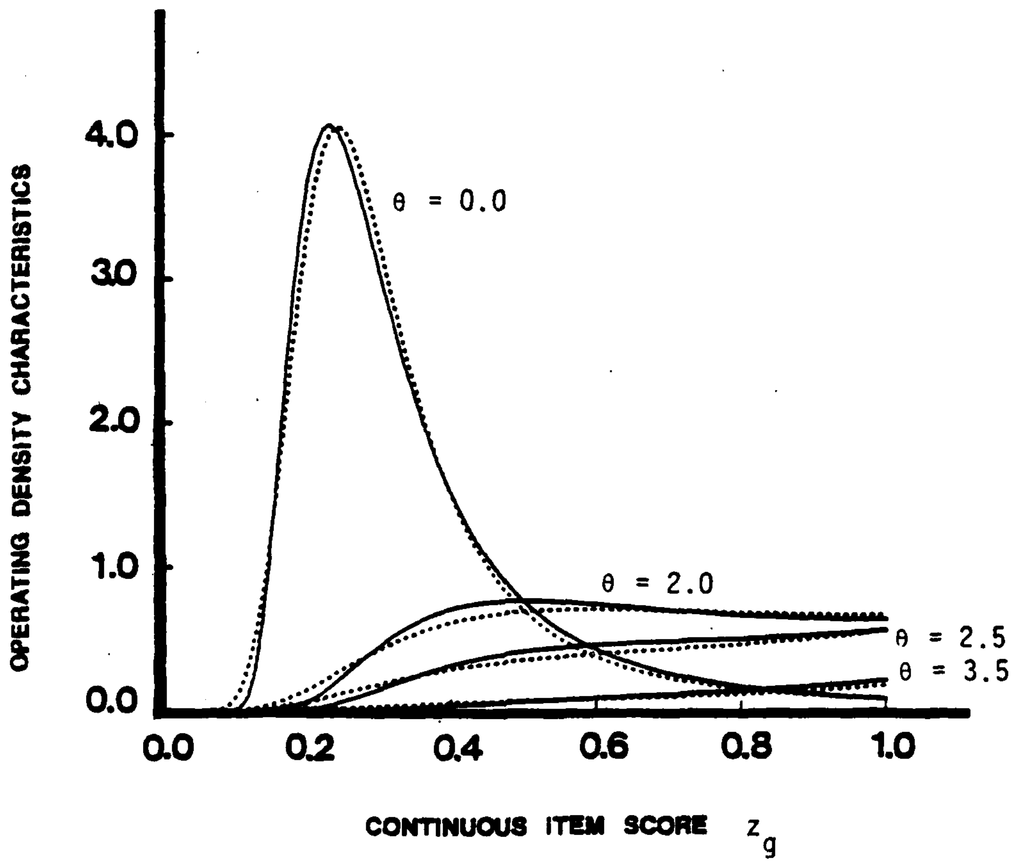

FIGURE 3-7 (Continued)

each of the thirteen fixed values of θ, i.e., -3.0, -2.5, -2.0, -1.5, -1.0, -0.5, 0.0, 0.5, 1.0, 1.5, 2.0, 2.5 and 3.5, with the parameters, $a_g = 1.0$ and $b_1 = 2.0$, using (3.23) as the functional relationship between z_g and b_{z_g} with the specification of $\zeta(z_g)$ by (3.25) for $k = 1$ and $\alpha_1 = 1.0$. As before, in the logistic model, 1.7 is used for the scaling factor D. Those graphs and four more sets of graphs with $\alpha_k = 1.0$ for $k = 3, 5, 7, 9$, respectively, are given in Appendix III as Figures A-3-1 through A-3-5, reversing the abscissa and the ordinate.

Formula (3.5) suggests that, in the open/closed response situation, the area under the curve of the operating density characteristic, $H_{z_g}(\theta)$, solely depends upon the value of $P_1(\theta)$, and again it is independent of the particular functional formula for the relationship between z_g and b_{z_g}. Thus, in each model, the areas shown in Figures A-3-1 through A-3-5 are equal, provided that the fixed values of θ are the same. The last two columns of Table 3-1 present those areas for the thirteen different values of θ in the normal ogive and the logistic models.

IV <u>Estimation of the Operating Density Characteristics</u>

Estimation of the operating density characteristics can be classified into two categories, one of which is the parametric estimation and the other the non-parametric estimation. In the parametric estimation, some specific model, like the normal ogive or the logistic model, is selected, and then the item parameters of the operating density characteristics are estimated. Sometimes the estimation is done concurrently with the estimation of the individual parameter θ_s of examinee s, and sometimes they are conducted separately. In any case, the estimation of the operating density characteristics is reduced to the item parameter estimation, and it

becomes relatively simple. If we fail to choose an appropriate model, or models, however, the whole research will become meaningless. For this reason, model validation is an important and necessary procedure in the parametric estimation, and it should be conducted at the end of each stage of research.

We notice that, in the parametric approach, we can always reduce the data based upon the continuous response level to those either upon the graded or upon the dichotomous response level, by categorizing the continuous responses into appropriate discrete response categories. Thus those methods developed for the item parameter estimation on the dichotomous response level (e.g., Lord, 1952, Bock and Aitkin, 1981) and their variations developed for the graded response level, are directly applicable in estimating the item parameters through $P^*_{z_g}(\theta)$ for selected values of z_g . By adopting an appropriate set of values of z_g , we shall be able to obtain the corresponding set of estimated b_{z_g}'s , and then by an appropriate curve fitting we shall be able to obtain the estimated difficulty parameter function. Since our data on the continuous response level contain more information, in so doing we can also conduct a model validation study, if we design our research appropriately.

In the non-parametric estimation of the operating density characteristics, we assume no mathematical forms a priori, and try to approach the operating density characteristics directly. Again, we can reduce our data to those which are based upon the graded response level, and those non-parametric methods developed for discrete responses (e.g., Levine, 1980, Samejima, 1977, 1981) can be applied, provided that we have <u>Old Test</u>, or a set of items whose operating characteristics or operating density characteristics are already known. When we do not have

Old Test, we can select a certain number of items which have high content validity out of all the items used in our research, and use this subset in place of the Old Test. In so doing, we may assume several different models for our "Old Test" items, estimate the item parameters using suitable parametric methods, conduct model validation studies for each model, and select a model which has the highest validity. We may end up with selecting different models for different items. In such a case, as far as each model satisfies the unique maximum condition (Samejima, 1969, 1972, 1973), we can still obtain the maximum likelihood estimate of the subject's latent trait, or individual parameter, by using the basic functions (Samejima, 1969, 1973) based upon those separate models.

In a half-open, half-closed response situation, or in the closed response situation, there is another conceivable method, which is a combination of the parametric approach and the non-parametric approach. In the first situation, we can reduce our data to those on the dichotomous response level by using the endpoint with a non-zero probability as one category, and all the other responses as the other category. Then we can use all the items thus dichotomized as the Old Test, searching a suitable model, or models, in the same way described in the preceding paragraph. In the second situation, we can reduce our data to those on the graded response level with three response categories, using both endpoints as the lowest and highest categories and all the other responses as the intermediate category, and follow the same procedure. We can also think of many variations of this new method, increasing the number of graded response categories by appropriate groupings.

The main difference between this new method and the preceeding one is that in the new method we make use of all the items used in our research as the Old Test, while in the other only a subset of items is used. In

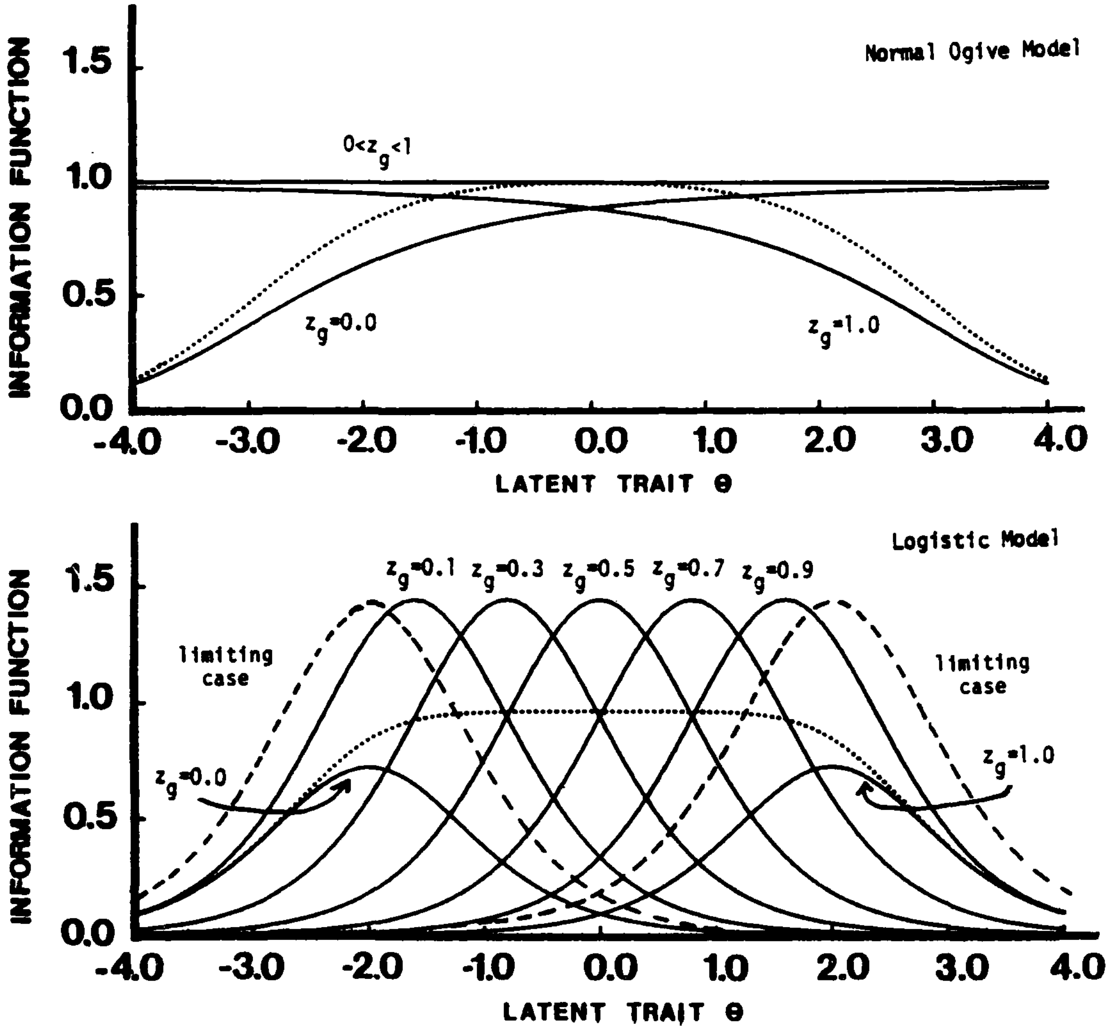

FIGURE 5-3

Item Response Information Functions, $I_{z_g}(\theta)$, (Solid Line) and Item Information Function, $I_g(\theta)$, (Dotted Line) in the Normal Ogive and the Logistic Models, with $a_g = 1.0$, $b_0 = -2.0$, $b_1 = 2.0$ and $D = 1.7$. In the Normal Ogive Model, the Horizontal Line Indicates Common $I_{z_g}(\theta)$ for All Item Scores , $0 < z_g < 1$, While in the Logistic Model the Five Curves Identical in Shape Indicate $I_{z_g}(\theta)$ for $z_g = 0.1, 0.3, 0.5, 0.7, 0.9$, When the Functional Relationship between z_g and b_{z_g} is Linear, with the Two Dashed Curves as Those in the Limiting Situations When z_g Tends to Zero and Unity, Respectively. Closed Response Situation.

model, so that we have

$$
(5.16) \qquad I_{z_g}(\theta) \begin{cases} = a_g^{\,2}\ \Psi_g\{a_g(\theta-b_0)\}[-a_g(\theta-b_0)\ Q_0^*(\theta) \\ \qquad + \Psi_g\{a_g(\theta-b_0)\}]\ [Q_0^*(\theta)]^{-2} & z_g = 0 \\ = a_g^{\,2} & 0 < z_g < 1 \\ = a_g^{\,2}\ \Psi_g\{a_g(\theta-b_1)\}[a_g(\theta-b_1)\ P_1^*(\theta) \\ \qquad + \Psi_g\{a_g(\theta-b_1)\}]\ [P_1^*(\theta)]^{-2} & z_g = 1\ . \end{cases}
$$

We can see from (5.16) that, for each and every value of z_g, the item response information function is positive for the entire range of θ. The uniqueness of the maximum likelihood estimate $\hat{\theta}_V$ is assured, therefore, for each and every response pattern V in the normal ogive model. In particular, the function assumes a constant, i.e., the square of the discrimination parameter a_g, for $0 < z_g < 1$. The upper graph of Figure 5-3 presents by solid lines the item response information function in the normal ogive model with the same parameter vaules used in Figures 5-1 and 5-2. The horizontal line in Figure 5-3 represents the item response information function for each and every continuous item score z_g which is greater than zero and less than unity.

In the logistic model, we obtain from (5.14) and (5.15) for the item response information function

$$
(5.17) \qquad I_{z_g}(\theta) \begin{cases} = D^2 a_g^{\,2}\ P_0^*(\theta)\ Q_0^*(\theta) & z_g = 0 \\ = 2D^2 a_g^{\,2}\ P_{z_g}^*(\theta)\ Q_{z_g}^*(\theta) & 0 < z_g < 1 \\ = D^2 a_g^{\,2}\ P_1^*(\theta)\ Q_1^*(\theta) & z_g = 1\ . \end{cases}
$$

Again, we can see that, for each and every value of z_g, the function is

situations, however, which include $z_g = 0$ or $z_g = 1$, or both, although we can still use $t(V)$ defined above for any response pattern which does not include either zero or unity as its elements. It is also recalled (Birnbaum, 1968) that a sufficient statistic, $t^*(V) = \sum_{u_g \varepsilon V} a_g u_g$, exists in the logistic model on the dichotomous response level where z_g is replaced by the binary item score u_g . Although the basic functions for $z_g = 0$ and $z_g = 1$ shown in (5.14) are identical with the corresponding functions for $u_g = 0$ and $u_g = 1$ on the dichotomous response level with the replacement of $P_0^*(\theta)$ by $P_1^*(\theta)$, a simple sufficient statistic does not exist, even though $t^*(V)$ can be used for any response pattern which solely consists of 0 and 1 . In general, the maximum likelihood estimation of the individual parameter must be conducted numerically through the basic functions for each response pattern.

For the item response information function, $I_{z_g}(\theta)$, we can write, in general,

$$(5.15) \qquad I_{z_g}(\theta) = -\frac{\partial^2}{\partial\theta^2} \log P_{z_g}(\theta) = -\frac{\partial}{\partial\theta} A_{z_g}(\theta)$$

$$\begin{cases} = a_g[(\frac{\partial}{\partial\theta}\Psi_g\{a_g(\theta-b_0)\})Q_0^*(\theta) \\ \quad + a_g(\Psi_g\{a_g(\theta-b_0)\})^2][Q_0^*(\theta)]^{-2} & z_g = 0 \\ = [-(\frac{\partial^2}{\partial\theta^2}\Psi_g\{a_g(\theta-b_{z_g})\})\Psi_g\{a_g(\theta-b_{z_g})\} \\ \quad + (\frac{\partial}{\partial\theta}\Psi_g\{a_g(\theta-b_{z_g})\})^2][\Psi_g\{a_g(\theta-b_{z_g})\}]^{-2} & 0 < z_g < 1 \\ = -a_g[(\frac{\partial}{\partial\theta}\Psi_g\{a_g(\theta-b_1)\})P_1^*(\theta) \\ \quad - a_g(\Psi_g\{a_g(\theta-b_1)\})^2][P_1^*(\theta)]^{-2} & z_g = 1 \ . \end{cases}$$

By virtue of (5.13) this function can be specified for the normal ogive

z_g , which are not illustrated here. Note that all these linear functions for $0 < z_g < 1$ are densely located between the two dashed lines representing the two limiting situations where z_g tends to zero and unity, respectively. The second graph of Figure 5-2 illustrates the basic functions in the logistic model with the same parameter values as those adopted in the normal ogive model, with the scaling factor, $D = 1.7$, for each of the seven item scores, $z_g = 0.0, 0.1, 0.3, 0.5, 0.7, 0.9, 1.0$. Again in this figure, the reader can imagine the curves for the basic functions for different values of z_g which are not drawn there, and all of which are densely located between the two dashed curves representing the two limiting situations where z_g tends to zero and unity, respectively.

It is obvious from (5.12) that, unlike the operating density characteristic, the basic function $A_{z_g}(\theta)$ does not depend upon the functional relationship between z_g and b_{z_g} , and those functions are identical for the same values of b_{z_g} regardless of the difficulty parameter function. In both the normal ogive and the logistic models, moreover, the basic functions for $0 < z_g < 1$ are identical in shape except for the positions on the abscissa, and, therefore, the functional relationship between z_g and b_{z_g} solely affects the position of each curve on the abscissa. The four sets of examples of basic functions resultant from the four different functional relationships between z_g and b_{z_g} , which are provided by the four solid curves in Figure 3-3, are given in Appendix V as Figure A-5, for both the normal ogive and the logistic models.

It should be recalled (Samejima, 1973, 1974) that a sufficient statistic, $t(V) = \sum_{z_g \in V} a_g^2 b_{z_g}$, exists in the normal ogive model in the open response situation. It is not the case with the other three response

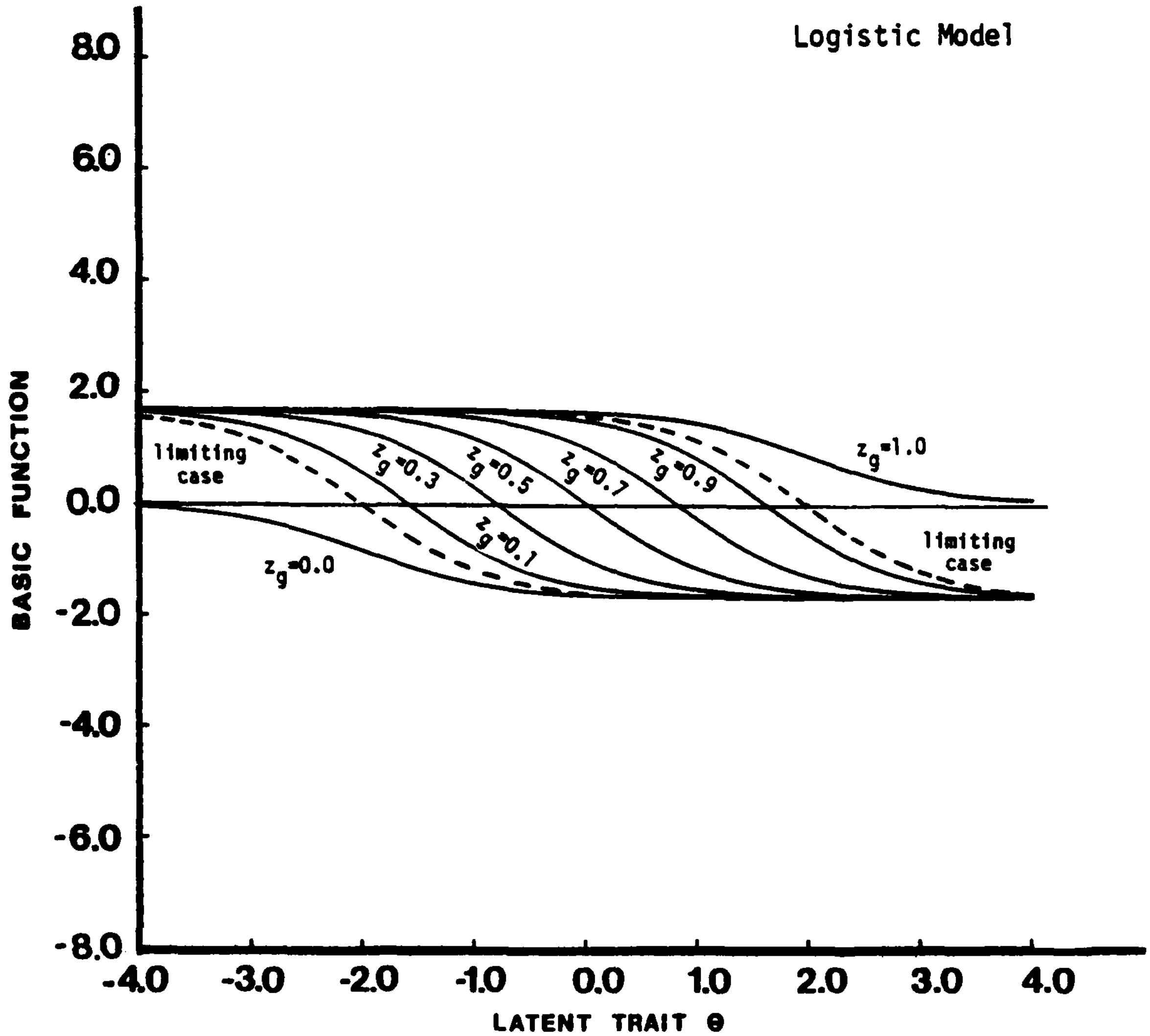

FIGURE 5-2 (Continued)

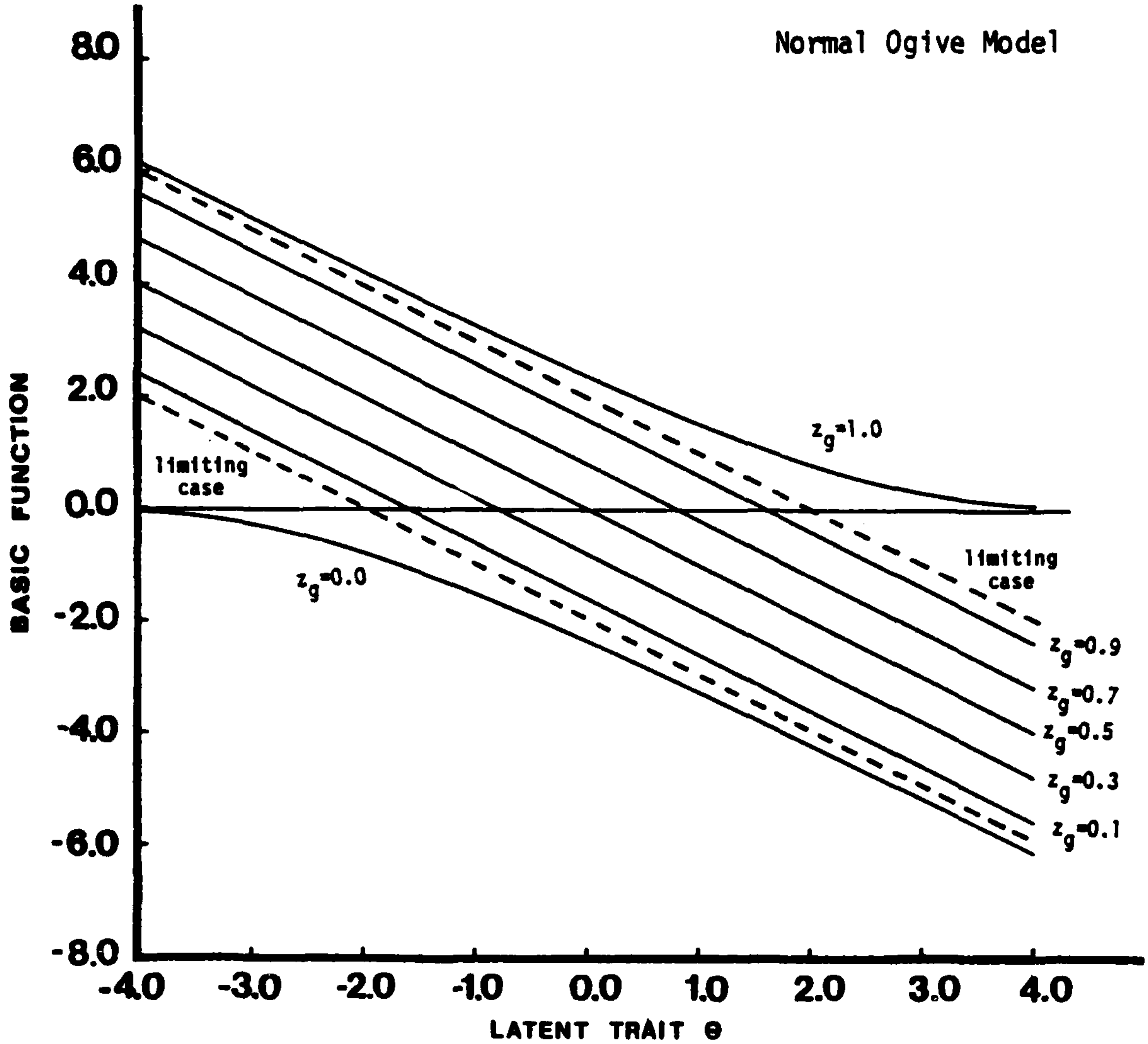

FIGURE 5-2

Basic Function $A_{z_g}(\theta)$, for Each of the Seven Values of the Item Score, 0.0 , 0.1 , 0.3 , 0.5 , 0.7 , 0.9 and 1.0 , Following the Normal Ogive and the Logistic Models, with $a_g = 1.0$, $b_0 = -2.0$, $b_1 = 2.0$ and $D = 1.7$, When the Linear Relationship Holds between the Item Score z_g and the Difficulty Parameter b_{z_g} . Closed Response Situation.

example, this proportion increases as z_g increases, and, in contrast, if it is given by one of the four dotted curves in Figure 3-3, the proportion decreases as z_g increases. The operating density characteristics for the first four cases are illustrated in Appendix IV as Figures A-4 for the same set of five values of z_g, in both the normal ogive and the logistic models. From (3.16), it is obvious that this proportion in the limiting situation where z_g tends to unity is $a_g \sum_{j=1}^{k} \alpha_j j$. In our example, therefore, this value equals k for $k = 3, 5, 7, 9$, and those curves are drawn by dashed lines in Figure A-4. It is also obvious from (3.16) that this proportion in the other limiting situation is zero, so those curves are degenerated to the line overlapping the abscissa and are not shown in Figure A-4. We notice that the corresponding sets of operating characteristics obtainable from the functional relationships given by dotted curves in Figure 3-3 are mirror images of those shown in Figure A-4 with $\theta = 0.0$ as the axis of rotation.

The examples shown so far for the closed response situation are relatively simple ones, in which $\frac{d}{dz_g} b_{z_g}$ is either constant or a monotone function of z_g . Formula (3.14) includes many other more complicated relationships, however, and we can conceive of varieties of different configurations of the curves for the operating density functions in sizes and in relative positions on the abscissa.

The first graph of Figure 5-2 presents the basic functions in the normal ogive model with the same item parameters, $a_g = 1.0$, $b_0 = -2.0$ and $b_1 = 2.0$, for the same set of selected item scores, $z_g = 0.1$, 0.3 , 0.5 , 0.7 , 0.9 , when b_{z_g} is the linear function of z_g given by (3.11), together with those for $z_g = 0.0$ and for $z_g = 1.0$. It will be easy for the reader to picture the basic functions for any other value of

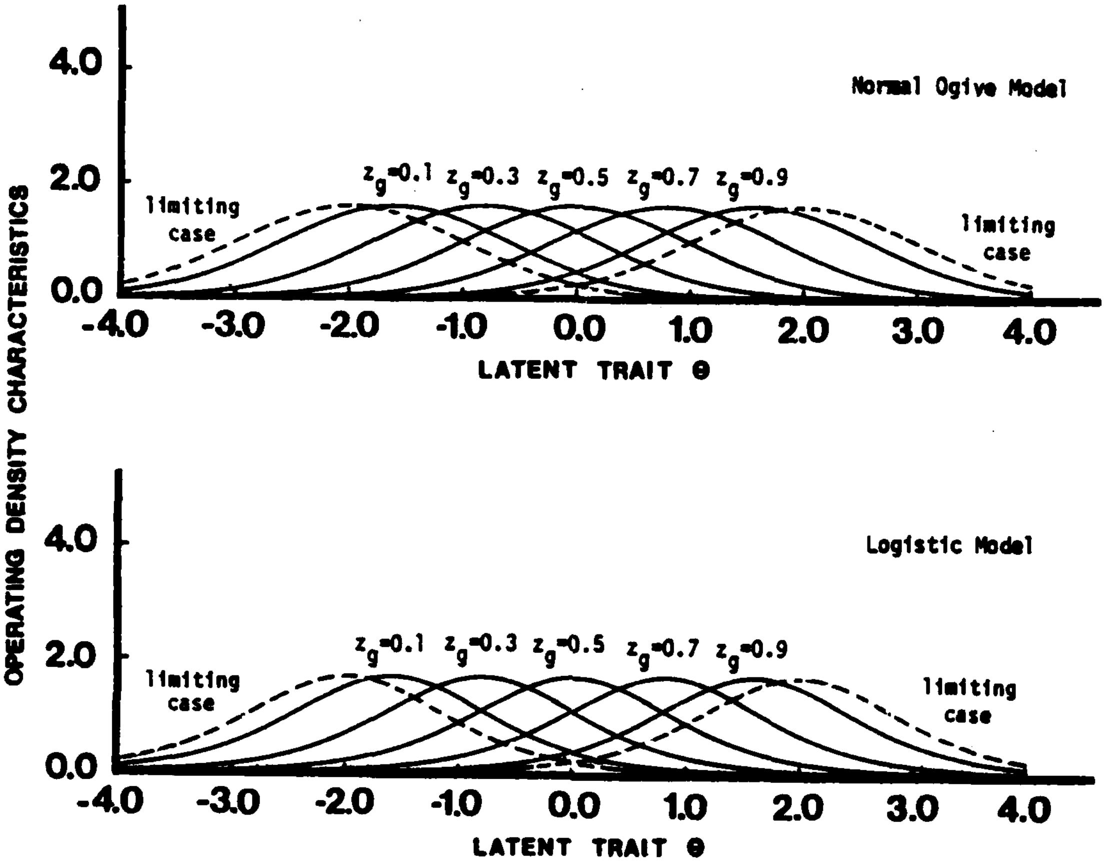

FIGURE 5-1

Operating Density Characteristic, $H_{z_g}(\theta)$, as a Function of θ for Each of the Five Values of the Item Score, 0.1 , 0.3 , 0.5 , 0.7 , and 0.9 , Following the Normal Ogive and the Logistic Models, with $a_g = 1.0$, $b_0 = -2.0$, $b_1 = 2.0$ and $D = 1.7$, When the Linear Relationship Holds between the Item Score z_g and the Difficulty Pa[illegible] b_{z_g} . The Additional Two Curves Are Those in the Limiting Situations Where z_g Tends to Zero and Unity, Respectively. Closed Response Situation.

The two asymptotes of the basic function are zero and $-Da_g$ for $z_g = 0$, Da_g and $-Da_g$ for $0 < z_g < 1$, and Da_g and zero for $z_g = 1$, respectively.

The upper graph of Figure 5-1 presents five examples of the operating density characteristic, $H_{z_g}(\theta)$, of the continuous item response z_g in the normal ogive model with the item parameters $a_g = 1.0$, $b_0 = -2.0$ and $b_1 = 2.0$, for $z_g = 0.1, 0.3, 0.5, 0.7, 0.9$ in the closed response situation, where the difficulty parameter b_{z_g} is given as the linear function of z_g. In the same graph, also presented by dashed lines are those in the two limiting cases where z_g tends to zero and unity, respectively. The corresponding five operating characteristics and those in the two limiting cases in the logistic model with the same set of item parameters and the scaling factor, $D = 1.7$, are shown in the lower graph of Figure 5-1. We can see that the results are very close to those in the normal ogive model.

As is obvious from (3.9), (3.10) and (3.13), in each model, those curves for the operating density characteristics for different values of z_g are identical except for the positions on the abscissa, and proportional by the ratio of $a_g(b_1-b_0)$ to the curve representing $\Psi_g(\cdot)$, i.e., the normal density or the logistic density function, with a_g^{-1} as the dispersion parameter and $b_0 + (b_1-b_0)z_g$ as the location parameter. Note, however, this is a special case in which b_{z_g} is the linear function of z_g. In general, although the curve for $H_{z_g}(\theta)$ is proportional to the one representing $\Psi_g(\cdot)$ with a_g^{-1} as the dispersion parameter and b_{z_g} as the location parameter, the ratio of the proportionality is $a_g(\frac{d}{dz_g} b_{z_g})$, which is a function of z_g. If the functional relationship between z_g and b_{z_g} is given by one of the four solid curves in Figure 3-3, for

between z_g and the item difficulty parameter b_{z_g} for each item g are known or well estimated.

In the normal ogive model, which is characterized by (3.9), we can see that the basic function takes the form

$$(5.13)\qquad A_{z_g}(\theta) \begin{cases} = -(2\pi)^{-1/2}a_g \exp[-a_g{}^2(\theta-b_0)^2/2]\ [Q_0^*(\theta)]^{-1} & z_g = 0 \\ = -a_g{}^2(\theta-b_{z_g}) & 0< z_g< 1 \\ = (2\pi)^{-1/2}a_g \exp[-a_g{}^2(\theta-b_1)^2/2]\ [P_1^*(\theta)]^{-1} & z_g = 1 \end{cases} .$$

This function is strictly decreasing in θ for all the values of z_g , and, in particular, for $0 < z_g < 1$ it is a linear function with the slope $-a_g{}^2$ which intercepts the abscissa at $\theta = b_{z_g}$. The two asymptotes of this basic function are zero and negative infinity for $z_g = 0$, positive and negative infinities for $0 < z_g < 1$, and positive infinity and zero for $z_g = 1$, respectively.

The basic function in the logistic model, which is specified by (3.10), is obtained from (5.12) and (3.10), so that we have

$$(5.14)\qquad A_{z_g}(\theta) \begin{cases} = -Da_g\ P_0^*(\theta) & z_g = 0 \\ = Da_g\ [1-2P_{z_g}^*(\theta)] & 0< z_g< 1 \\ = Da_g\ Q_1^*(\theta) & z_g = 1 \end{cases} .$$

We can see that this basic function in the logistic model is also strictly decreasing in θ throughout the entire range of θ for each and every item score z_g , as is the case with the normal ogive model. Unlike in the normal ogive model, however, it is not a linear function for $0< z_g< 1$, although it intercepts the abscissa at the same point, i.e., $\theta = b_{z_g}$.

$$(5.11) \qquad \frac{\partial}{\partial\theta} \log L_V(\theta) = \sum_{z_g \varepsilon V} \frac{\partial}{\partial\theta} \log P_{z_g}(\theta) \equiv 0 ,$$

treating z_g as if it had a discrete distribution. We notice that the term under the summation sign in this equation is the basic function $A_{z_g}(\theta)$ of the item response z_g. From this and (5.9), (3.1), (3.3), (3.7) and (3.8) we can write for the general form of the basic function in the closed response situation

$$(5.12) \qquad A_{z_g}(\theta) \begin{cases} = -a_g \ \Psi_g\{a_g(\theta-b_0)\}[Q_0^*(\theta)]^{-1} & z_g = 0 \\ = [\frac{\partial}{\partial\theta} \ \Psi_g\{a_g(\theta-b_{z_g})\}][\Psi_g\{a_g(\theta-b_{z_g})\}]^{-1} & 0 < z_g < 1 \\ = a_g \ \Psi_g\{a_g(\theta-b_1)\}[P_1^*(\theta)]^{-1} & z_g = 1 \end{cases} ,$$

whose middle line is identical with (5.2). In the closed response situation, and also in the closed/open and the open/closed response situations, therefore, we can see that the basic function $A_{z_g}(\theta)$ given by (5.2) is also valid for any value of z_g which satisfies $0 < z_g < 1$.

We can see from (5.12) that for $z_g = 0$ the basic function is negative and for $z_g = 1$ it is positive, throughout the entire range of θ except, at most, at an enumerable number of points. It has been shown (Samejima, 1972) in a somewhat different context that, if $\Psi_g(\cdot)$ follows one of the formulae such as (3.9) and (3.10), those three functions in (5.12) are strictly decreasing in θ, the fact that leads to the unique maximum for the likelihood function $L_V(\theta)$ for each and every response pattern V. The unique maximum likelihood estimate of the individual parameter of the subject whose response pattern is V can be obtained through (5.11) and (5.12), therefore, provided that the item discrimination parameter a_g for each item g and the functional relationship

$$\text{(5.7)} \qquad \tilde{A}_{x_g}(\theta) = \frac{\partial}{\partial\theta} \log \left[\frac{\partial}{\partial\theta} P^*_{x_g}(\theta)\right]$$

$$= \left[\frac{\partial}{\partial\theta} \Psi_g\{a_g(\theta-b_{x_g})\}\right]\left[\Psi_g\{a_g(\theta-b_{x_g})\}\right]^{-1} ,$$

which is identical with (5.2) with the replacement of the item response difficulty parameters b_{x_g} by b_{z_g} .

An alternative way of deriving (5.2) in the framework of the closed response situation may be as follows. Since we can write for the operating density characteristic, $H_{z_g}(\theta)$, as the limiting situation such that

$$\text{(5.8)} \qquad H_{z_g}(\theta) = \lim_{\Delta z_g \to 0} \frac{P^*_{z_g}(\theta) - P^*_{(z_g+\Delta z_g)}(\theta)}{\Delta z_g}$$

by using z_g and Δz_g instead of y_g and Δy_g , we could replace $P_{z_g}(\theta)$ for any value of z_g between zero and unity by $H_{z_g}(\theta)\, dz_g$. From this and (3.7), we can write

$$\text{(5.9)} \qquad P_{z_g}(\theta) \begin{cases} = Q^*_0(\theta) & z_g = 0 \\ = H_{z_g}(\theta)\, dz_g & 0 < z_g < 1 \\ = P^*_1(\theta) & z_g = 1 \end{cases} .$$

The likelihood function, $L_V(\theta)$, for the response pattern V , or the vector of n item responses, is given by

$$\text{(5.10)} \qquad L_V(\theta) = \prod_{z_g \in V} P_{z_g}(\theta) \quad ,$$

by virtue of the conditional independence of the n item scores z_g , given θ . From (5.9) and (5.10) we can write for the likelihood equation

homogeneous case of the graded response level. To follow this logic, it may be helpful to go back to the rationale which leads to the definition of $H_{z_g}(\theta)$ (cf. Samejima, 1973). Let y_g be the relative graded item score, which is defined by

$$(5.3)\qquad y_g = \frac{x_g}{m_g} = 0\ ,\ \frac{1}{m_g}\ ,\ \frac{2}{m_g}\ ,\ \dots\ ,\ \frac{m_g-1}{m_g}\ ,\ 1\ ,$$

and Δy_g be the increment in y_g , so that

$$(5.4)\qquad \Delta y_g = \frac{1}{m_g}\ .$$

Thus we can consider the continuous item response z_g as the limiting situation of the <u>relative graded item score</u> y_g when Δy_g tends to zero. The operating characteristic, $P_{x_g}(\theta)$, of the graded item response x_g to item g is given by

$$(5.5)\qquad \begin{aligned} P_{x_g}(\theta) &= P^*_{x_g}(\theta) - P^*_{(x_g+1)}(\theta) \\ &= P^*_{y_g}(\theta) - P^*_{(y_g+\Delta y_g)}(\theta) \\ &= P_{y_g}(\theta) \end{aligned}$$

where

$$(5.6)\qquad P^*_{y_g}(\theta) = \int_{-\infty}^{a_g(\theta-b_{x_g})} \Psi_g(t)\,dt = P^*_{x_g}(\theta)\ ,$$

which is the counterpart of (3.1) on the continuous response level. The asymptotic basic function is defined by

general, the choice of a method should depend upon the nature of our data, including the configuration of the characteristics of our items, the sample size of subjects, and so forth.

V Estimation of the Latent Trait or Individual Parameter

When the item parameters are known, or well estimated, the estimation of the individual parameter, or the point of the latent trait θ at which the subject is located, can be performed through the maximum likelihood estimation. It is well-known that, when a simple sufficient statistic for the response pattern V or a sequence of n item scores which is given by

$$(5.1) \qquad V = (z_1, z_2, \dots, z_g, \dots, z_n)'$$

exists, this process of maximum likelihood estimation becomes relatively simple and straightforward. If a simple sufficient statistic does not exist, however, we will have to use the basic function (Samejima, 1969, 1972, 1973) and follow the numerical process to obtain the maximum likelihood estimate $\hat{\theta}_V$ for each response pattern V.

It has been shown (Samejima, 1973) that the basic function, $A_{z_g}(\theta)$, in the open response situation is given by

$$(5.2) \qquad A_{z_g}(\theta) = \frac{\partial}{\partial\theta} \log H_{z_g}(\theta) = [\frac{\partial}{\partial\theta} \Psi_g\{a_g(\theta-b_{z_g})\}][\Psi_g\{a_g(\theta-b_{z_g})\}]^{-1}$$

for any value of item response z_g which is greater than zero and less than unity. It has also been pointed out that this function is the same as the asymptotic basic function, $\tilde{A}_{x_g}(\theta)$, (Samejima, 1972) defined in the

positive throughout the entire range of θ , and the unique maximum condition is satisfied. Unlike in the normal ogive model, however, it does not assume a constant for any z_g between zero and unity, and, in fact, it indicates a unimodal curve with the modal point at $\theta = b_{z_g}$. The lower graph of Figure 5-3 presents the item response information function in the logistic model with the same set of parameter values and the scaling factor that we used in Figures 5-1 and 5-2 for $z_g = 0.0, 0.1, 0.3, 0.5, 0.7, 0.9, 1.0$, when the functional relationship between z_g and b_{z_g} is linear. In the same graph, those in the two limiting situations where z_g tends to zero and unity, respectively, are drawn by dashed lines. Again, the functional relationship between z_g and b_{z_g} solely affects the position of the curve for $I_{z_g}(\theta)$ on the abscissa, as is expected from the basic functions in the logistic model. The four sets of results obtained from the functional relationships between z_g and b_{z_g} which are provided by the four solid curves in Figure 3-3 are given in Appendix VI as Figure A-6. Note that they are for the logistic model only.

The item information function, $I_g(\theta)$, is defined as the conditional mean of the item response information function, given θ (Samejima, 1969, 1972, 1973), for which we can write

$$(5.18) \qquad I_g(\theta) = I_0(\theta)[1-P_0^*(\theta)] + \int_0^1 I_{z_g}(\theta)\, H_{z_g}(\theta)\, dz_g + I_1(\theta)\, P_1^*(\theta) ,$$

where $I_0(\theta)$ and $I_1(\theta)$ indicate the item response information function $I_{z_g}(\theta)$ for $z_g = 0$ and $z_g = 1$, respectively. This function is drawn by a dotted line in each graph of Figures 5-3 and A-6.

We have observed so far the basic function and the item response information function in the closed response situation, and specified the

formulae for the normal ogive and the logistic models. It should be noted that those formulae in the open response situation (Samejima, 1973) are included by the corresponding set of formulae in the closed response situation, for we can obtain the former set simply by adopting the middle formulae of (5.12) through (5.17) , ignoring the first and third ones. Similarly, in the closed/open and the open/closed response situations, we can take the first two and the last two formulae of each of the six sets of formulae, (5.12) through (5.17), ignoring the third and the first ones, respectively. Thus (5.12) through (5.17) represent the basic functions and the item response information functions in all the four response situations. As for the item information function, $I_g(\theta)$, we notice that the middle term of the right hand side of (5.18) provides us with the item information function in the open response situation. In the closed/open and the open/closed response situations, it is given by the first two and the last two terms of the right hand side of (5.18), ignoring the last and the first terms, respectively. Thus we can write

$$(5.19) \qquad I_g(\theta) = I_0(\theta)\{1-P_0^*(\theta)\} + \int_0^1 I_{z_g}(\theta)\, H_{z_g}(\theta)\, dz_g$$

for the closed/open response situation, and

$$(5.20) \qquad I_g(\theta) = \int_0^1 I_{z_g}(\theta)\, H_{z_g}(\theta)\, dz_g + I_1(\theta)\, P_1^*(\theta)$$

for the open/closed response sitaution.

Figure 5-4 illustrates the operating density characteristics $H_{z_g}(\theta)$ in the normal ogive and the logistic models in the closed/open response situation, with the item parameters $a_g = 1.0$ and $b_0 = -2.0$, and the

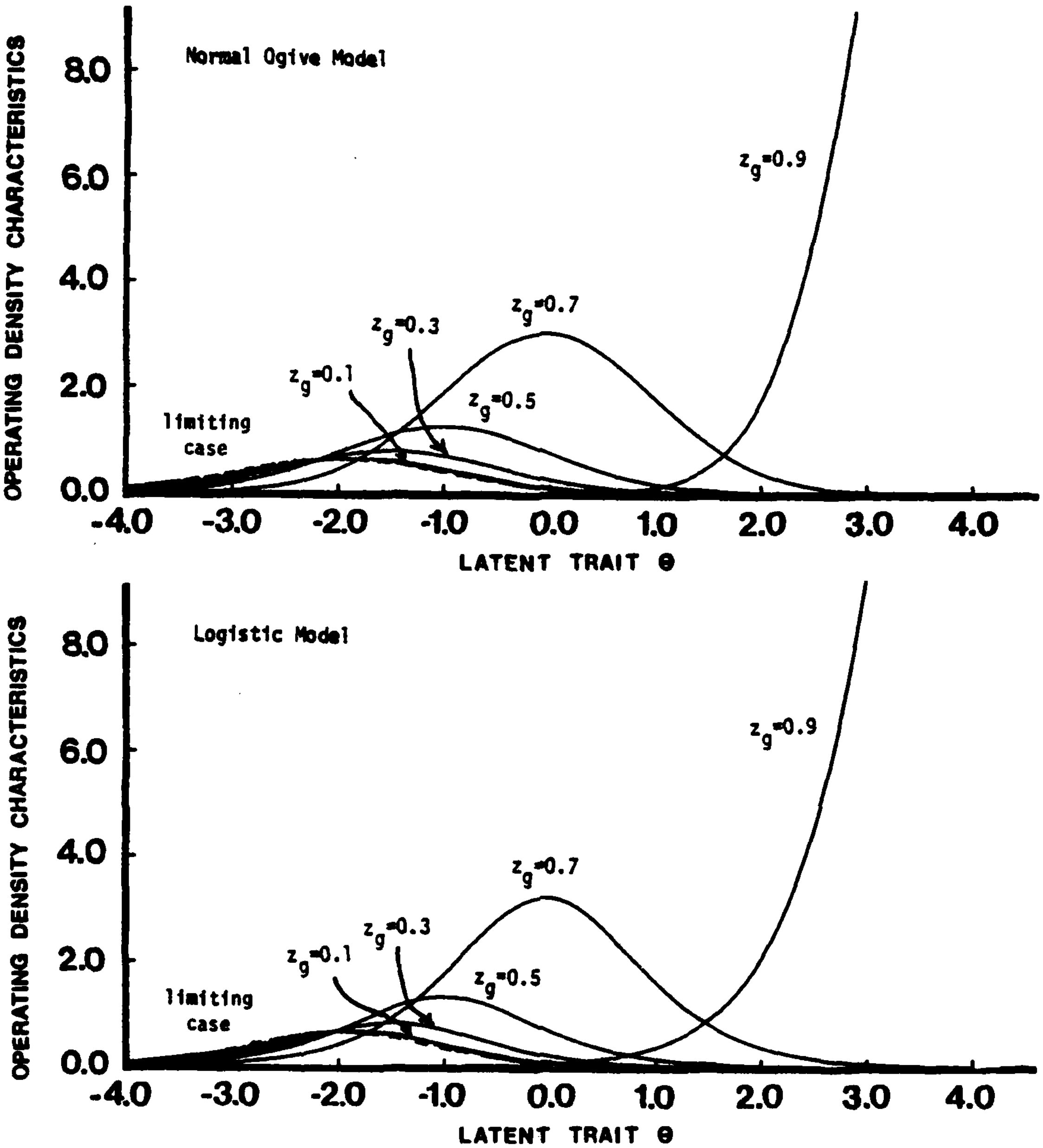

FIGURE 5-4

Operating Density Characteristic, $H_{z_g}(\theta)$, As a Function of θ for Each of the Five Values of the Item Score, 0.1 , 0.3 , 0.5 , 0.7 , and 0.9 , Following the Normal Ogive and the Logistic Models, with $a_g = 1.0$, $b_0 = -2.0$ and $D = 1.7$, When the Functional Relationship between the Item Score z_g and the Difficulty Parameter b_{z_g} Is Given by $b_{z_g} = b_0 + \tan[(\pi/2)z_g]$. The Additional Curve Is the One in the Limiting Situation Where z_g Tends to Zero. Closed/Open Response Situation.

scaling factor $D = 1.7$ in the latter model, for the five selected item scores, 0.1, 0.3, 0.5, 0.7, and 0.9 . The difficulty parameter function adopted here is given by

$$(5.21) \qquad b_{z_g} = b_0 + \tan [(\pi/2)z_g] ,$$

which is shown in Figure 3-4 as the solid curve marked with $k = 1$. As was observed in the closed response situation, this operating density characteristic is proportional to $\Psi_g(\cdot)$ with a_g^{-1} as the dispersion parameter and b_{z_g} as the location parameter with $a_g(\frac{d}{dz_g} b_{z_g})$ as the ratio of proportionality. Since in this example the derivative of the difficulty parameter function is given by $(\pi/2) \sec^2[(\pi/2)z_g]$ and it increases with z_g , the area under the curve of $H_{z_g}(\theta)$ in Figure 5-4 increases as z_g grows larger, both in the normal ogive and in the logistic model. In fact, the area approaches infinity as z_g tends to unity and, therefore, b_{z_g} tends to infinity, the tendency that is hinted by the truncated curves for $H_{z_g}(\theta)$ for $z_g = 0.9$ in the two graphs of Figure 5-4. On the other hand, when the continuous item score z_g tends to zero and, therefore, b_{z_g} tends to b_0 , the ratio of proportionality approaches $(\pi/2)a_g$, and this limiting case of $H_{z_g}(\theta)$ is shown by a dashed curve in Figure 5-4 in each of the normal ogive and the logistic models. The areas under the curves for the same value of z_g across the two graphs of Figure 5-4 are equal. Similar sets of six curves for the operating density characteristics $H_{z_g}(\theta)$ both in the normal ogive and in the logistic model are given in Appendix VII as Figure A-7 with the difficulty parameter function,

(5.22) $$b_{z_g} = b_0 + \tan[(\pi/2)\, z_g{}^k] ,$$

for $k = 3, 5, 7, 9$, respectively, which are also shown by solid curves in Figure 3-4. Since the derivative of this function is given by

(5.23) $$\frac{d}{dz_g} b_{z_g} = (\pi/2)\, k\, z_g{}^{k-1} \sec^2[(\pi/2)\, z_g{}^k] ,$$

the area under the curve for $H_{z_g}(\theta)$ decreases with k for a fixed value of z_g, as we can see in Figure A-7. Unlike the case where $k = 1$, which is shown in Figure 5-4, the ratio of proportionality approaches zero as z_g tends to zero, so the dashed curve for the limiting case is degenerated to a line overlapping the abscissa in each of the eight graphs of Figure A-7.

Figure 5-5 presents the basic function $A_{z_g}(\theta)$ for each of the six values of the item score, 0.0, 0.1, 0.3, 0.5, 0.7 and 0.9, in the normal ogive model and in the logistic model, respectively, with $a_g = 1.0$, $b_0 = -2.0$ and $D = 1.7$, when the difficulty parameter function is given by (5.21), in the closed/open response situation. As we can see from the middle lines of (5.13) and (5.14), and (5.21), when z_g tends to zero the basic function approaches $-a_g{}^2(\theta-b_0)$ and $Da_g[1-2P_0^*(\theta)]$ in the normal ogive model and in the logistic model, respectively, and those curves are drawn by dashed lines in Figure 5-5. In the other limiting case where z_g tends to unity, $A_{z_g}(\theta)$ approaches positive infinity in the normal ogive model and Da_g in the logistic model. Similar sets of six curves for the basic function $A_{z_g}(\theta)$ are given in Appendix VIII as Figure A-8, for the four cases which correspond to those shown in Appendix VII, in both the normal ogive and the logistic models. As is expected from the middle lines of (5.13) and (5.14), and (5.22), for any fixed value of z_g, as k

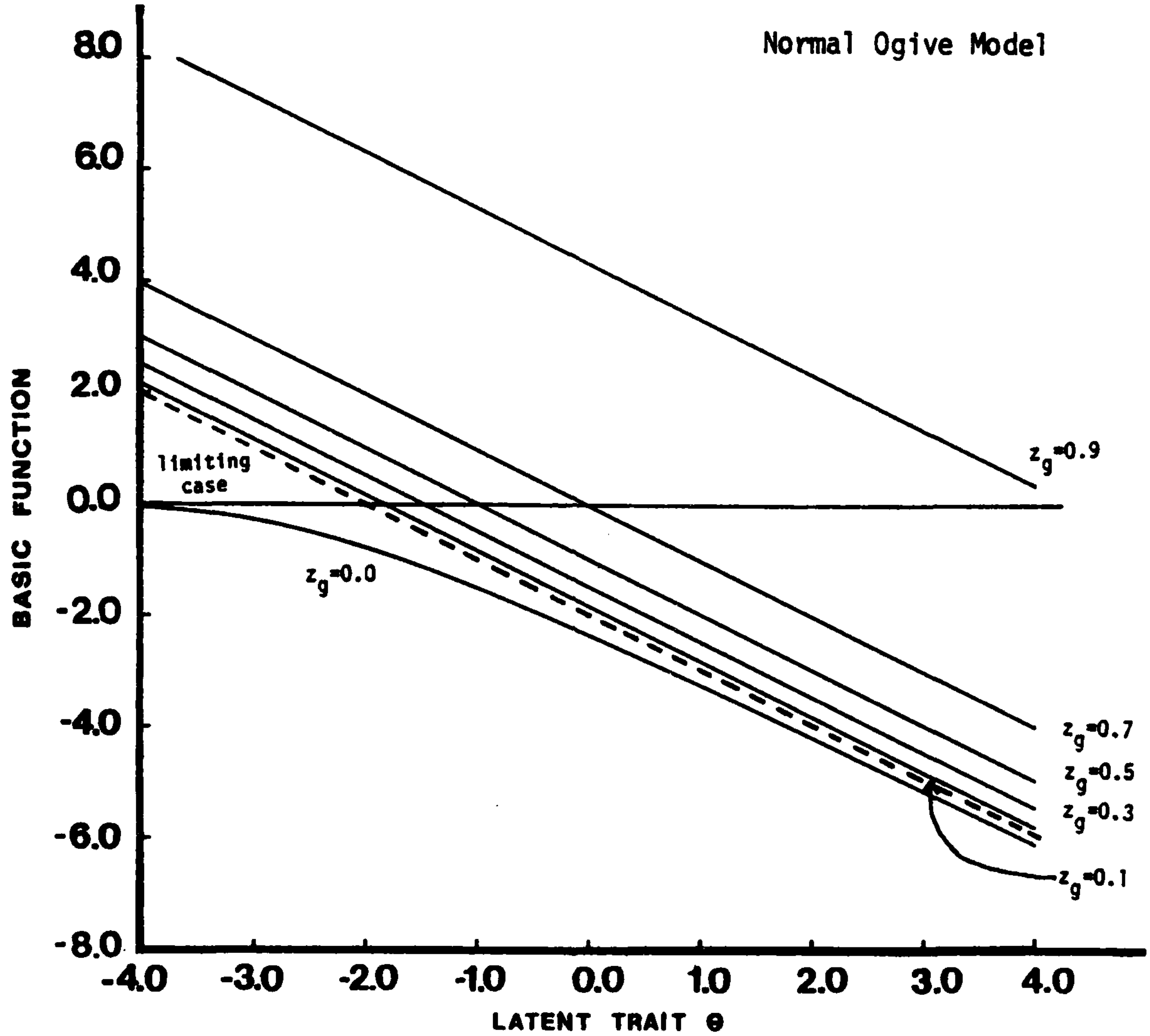

FIGURE 5-5

Basic Function, $A_{z_g}(\theta)$, for Each of the Six Values of the Item Score, 0.0 , 0.1 , 0.3 , 0.5 , 0.7 and 0.9 , Following the Normal Ogive and the Logistic Models, with $a_g = 1.0$, $b_0 = -2.0$ and $D = 1.7$, When the Functional Relationship between the Item Score z_g and the Difficulty Parameter b_{z_g} Is Given by $b_{z_g} = b_0 + \tan[(\pi/2)z_g]$. Closed/Open Response Situation.

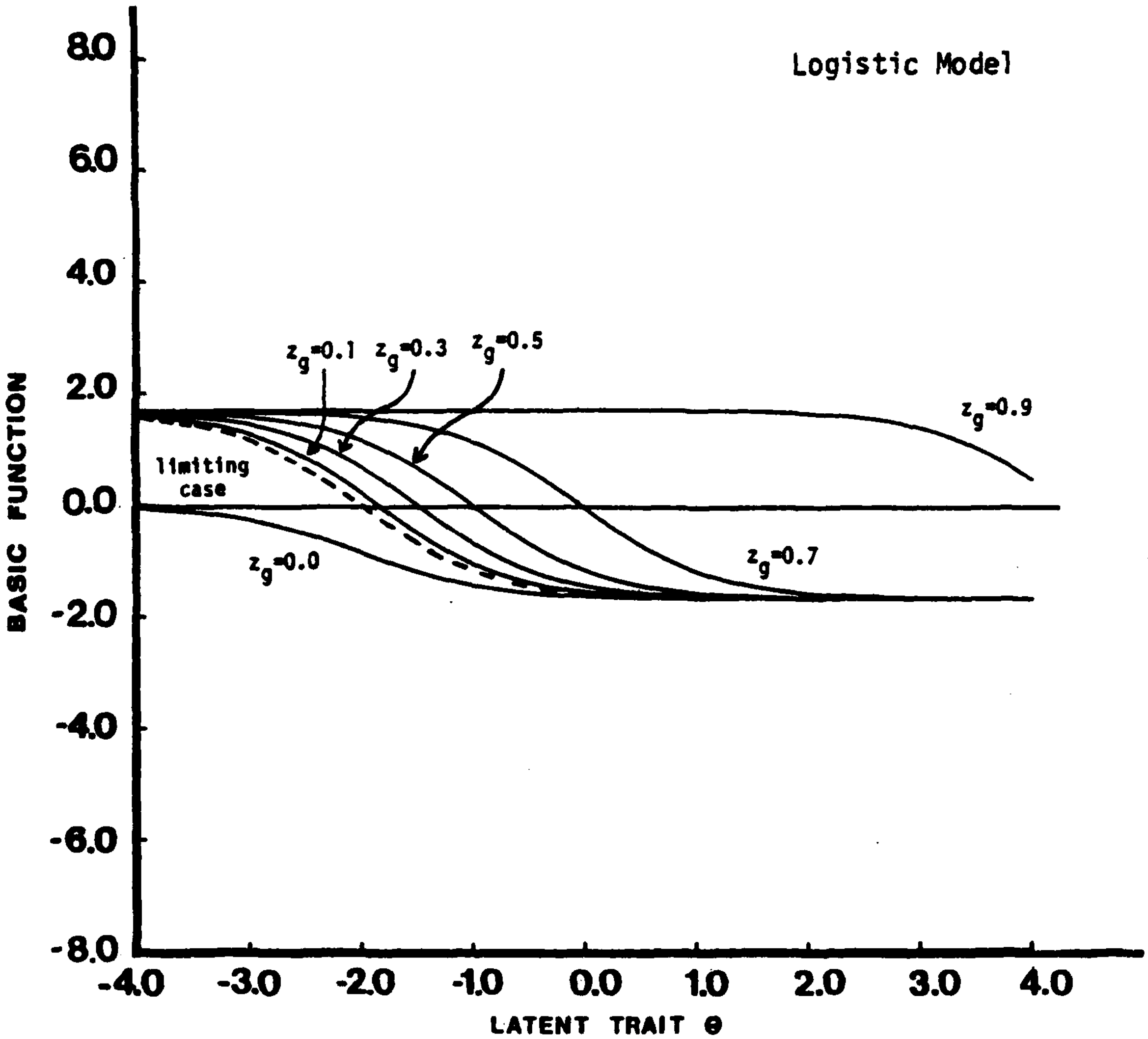

FIGURE 5-5 (Continued)

grows larger the curves get closer to the one for the limiting case where z_g tends to zero, and sometimes they are indistinguishable in vision from the dashed curve in the limiting case.

Figure 5-6 presents the item response information function $I_{z_g}(\theta)$ by solid lines and the item information function $I_g(\theta)$ by a dotted line in each of the normal ogive and the logistic models, with the same parameters, scaling factor, difficulty parameter function and fixed values of z_g as were used in Figures 5-4 and 5-5, together with the limiting case of $I_{z_g}(\theta)$ where z_g tends to zero which is drawn by a dashed line. We can see from (5.16) that in the normal ogive model the horizontal line in the upper graph of Figure 5-6 indicates the item response information function for each and every value of z_g in the interval $(0,1)$, so this includes the five cases where $z_g = 0.1, 0.3, 0.5, 0.7, 0.9$, and the one in the limiting case where z_g tends to zero also overlaps this line. In the logistic model, those six curves are separated, but identical in shape. In both models the item response information for $z_g = 0.0$ is less than the one in the limiting case where z_g tends to zero, the result which stems from the fact that this item score deals with the discrete part of the conditional item score distribution. We notice that the upper graph of Figure 5-6 is valid regardless of the differences in the difficulty parameter function that we adopt. The same is not true with the lower graph, however, since the item response information function in the logistic model depends upon the difficulty parameter b_{z_g}, as is obvious from the middle line of (5.17). The four sets of results in the logistic model using the same four difficulty parameter functions, that were used in both Appendices VII and VIII, are shown in Appendix IX as Figure A-9.

In the open/closed response situation, those characteristics are

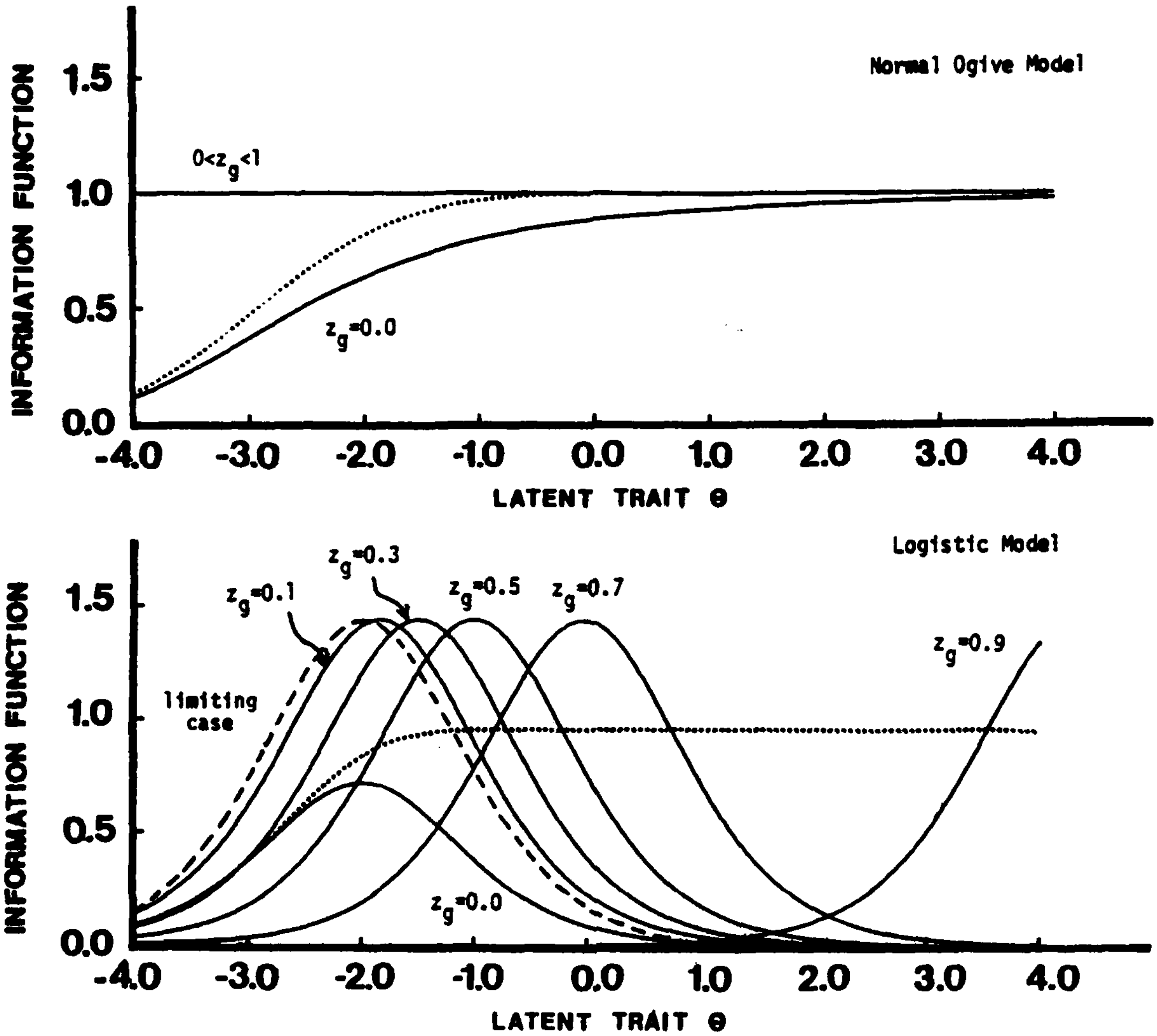

FIGURE 5-6

Item Response Information Functions, $I_{z_g}(\theta)$, (Solid Line) and Item Information Function, $I_g(\theta)$, (Dotted Line) in the Normal Ogive and the Logistic Models, with $a_g = 1.0$, $b_0 = -2.0$ and $D = 1.7$. In the Normal Ogive Model, the Horizontal Line Indicates Common $I_{z_g}(\theta)$ for All Item Scores, $0 < z_g < 1$, While in the Logistic Model the Five Curves Identical in Shape Indicate $I_{z_g}(\theta)$ for $z_g = 0.1, 0.3, 0.5, 0.7, 0.9$, When the Functional Relationship between z_g and b_{z_g} Is Given by $b_{z_g} = b_0 + \tan[(\pi/2)z_g]$, with the Dashed Curve as the One in the Limiting Situation Where z_g Tends to Zero. Closed/Open Response Situation.

more or less reversed. Figure 5-7 illustrates the operating characteristics $H_{z_g}(\theta)$ in the normal ogive and the logistic models in the open/closed situation, with the item parameters $a_g = 1.0$ and $b_1 = 2.0$, and the scaling factor $D = 1.7$ in the latter model, for the five item scores, 0.1, 0.3, 0.5, 0.7 and 0.9. The difficulty parameter function adopted in this illustration is given by

$$(5.24) \qquad b_{z_g} = b_1 + \tan\,[(-\pi/2)(1-z_g)^k]$$

for $k = 1$, which is shown in Figure 3-6 as one of the five solid curves. Again those operating density characteristics are proportional to $\Psi_g(\cdot)$ with a_g^{-1} as the dispersion parameter and b_{z_g} as the location parameter with $a_g(\frac{d}{dz_g} b_{z_g})$ as the ratio of proportionality. We can see from (5.24) that the derivative of the difficulty parameter function in this example is given by

$$(5.25) \qquad \frac{d}{dz_g} b_{z_g} = (\pi/2)\ k\ (1-z_g)^{k-1}\ \sec^2[(-\pi/2)(1-z_g)^{k-1}] \quad .$$

It is obvious from (5.25) that this derivative decreases with z_g, having the two asymptotes, i.e., positive infinity and $(\pi/2)$ for $k = 1$ and positive infinity and zero for $k > 1$, when z_g tends to zero and unity, respectively. Thus the ratio of proportionality also decreases with z_g, as we can see in Figure 5-7. Note that the area under each curve represents this ratio of proportionality, as is the case with those in the closed response and the closed/open response situations. In the same figure also presented by dashed lines are the operating densit, characteristics in the limiting case where z_g tends to unity, whose

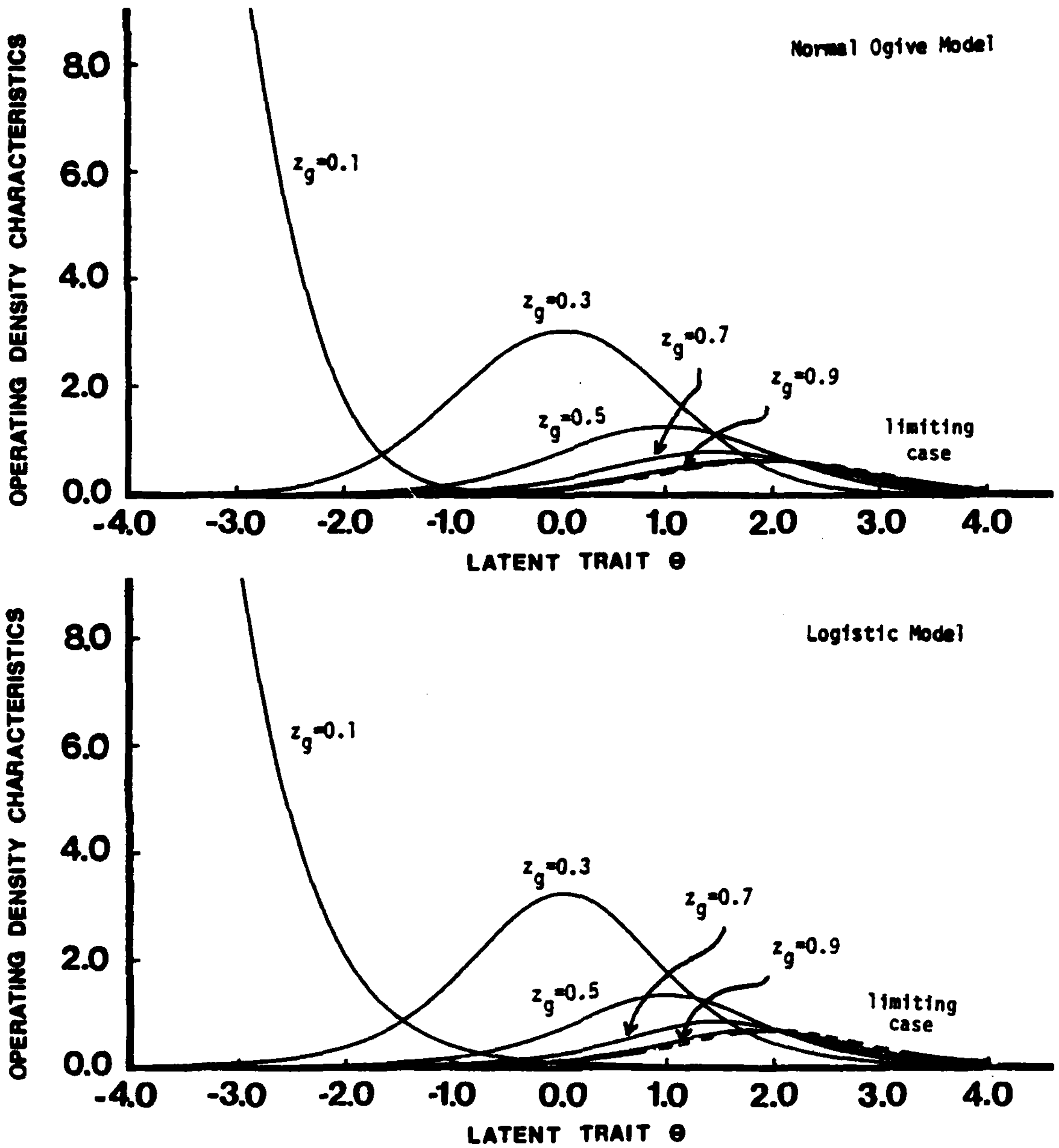

FIGURE 5-7

Operating Density Characteristic, $H_{z_g}(\theta)$, As a Function of θ for Each of the Five Values of the Item Score, 0.1, 0.3, 0.5, 0.7, and 0.9, Following the Normal Ogive and the Logistic Models, with $a_g = 1.0$, $b_1 = 2.0$ and $D = 1.7$, When the Functional Relationship between the Item Score z_g and the Difficulty Parameter b_{z_g} Is Given by $b_{z_g} = b_1 + \tan[(-\pi/2)(1-z_g)]$. The Additional Curve Is the One in the Limiting Situation Where z_g Tends to Unity. Open/Closed Response Situation.

area equals $(\pi/2)$, in the normal ogive and the logistic models. The corresponding sets of six curves for the operating density characteristics $H_{z_g}(\theta)$ are shown in Appendix X as Figure A-10, with the difficulty parameter function given by (5.24) for $k = 3, 5, 7, 9$, respectively, in both the normal ogive model and the logistic model. Since in those cases $H_{z_g}(\theta)$ in the limiting case where z_g tends to unity is degenerated to the line overlapping the abscissa, the dashed curves are not visible in those eight graphs. From (5.25) we can find that the ratio of proportionality of the curve decreases with k for a fixed value of z_g , as is shown in Figure A-10.

The basic function $A_{z_g}(\theta)$ for each of the six values of z_g is shown in Figure 5-8 in each of the normal ogive model and the logistic model, with the same item parameters, scaling factor and difficulty parameter function that were adopted in Figure 5-7, in the open/closed response situation. In the same figure, also presented by dashed lines are the basic functions in the limiting case where z_g tends to unity, i.e., $-a_g^2(\theta-b_1)$ and $Da_g[1-2P_1^*(\theta)]$, in the two models, respectively. In the other limiting case where z_g tends to zero, $A_{z_g}(\theta)$ approaches negative infinity in the normal ogive model and $-Da_g$ in the logistic model, as is obvious from the middle lines of (5.13) and (5.14). The corresponding sets of six solid curves for the basic function $A_{z_g}(\theta)$ and the dashed curve for the limiting case are drawn in Figure A-11 of Apendix XI, for the four cases where the difficulty parameter function is given by (5.24) with $k = 3$, 5 , 7 , 9 , respectively, in both the normal ogive and the logistic models. Dashed curves are practically invisible in these eight graphs because of their closeness to one or more curves for $A_{z_g}(\theta)$ for larger values of z_g .

Figure 5-9 presents the item response information function $I_{z_g}(\theta)$

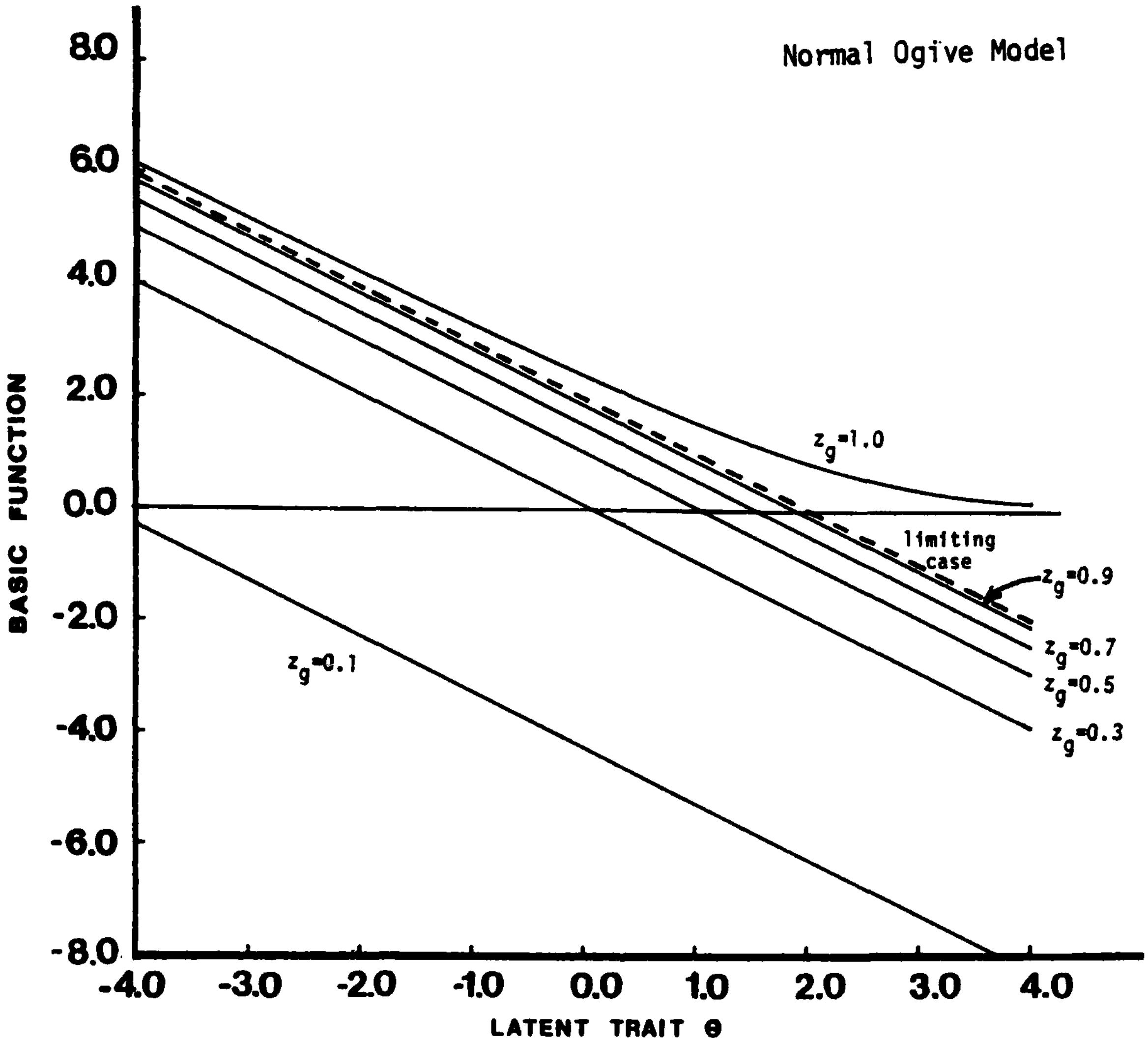

FIGURE 5-8

Basic Function, $A_{z_g}(\theta)$, for Each of the Six Values of the Item Score, 0.1, 0.3, 0.5, 0.7, 0.9 and 1.0, Following the Normal Ogive and the Logistic Models, with $a_g = 1.0$, $b_1 = 2.0$ and $D = 1.7$, When the Functional Relationship between the Item Score z_g and the Difficulty Parameter b_{z_g} Is Given by $b_{z_g} = b_1 + \tan[(-\pi/2)(1-z_g)]$. Open/Closed Response Situation.

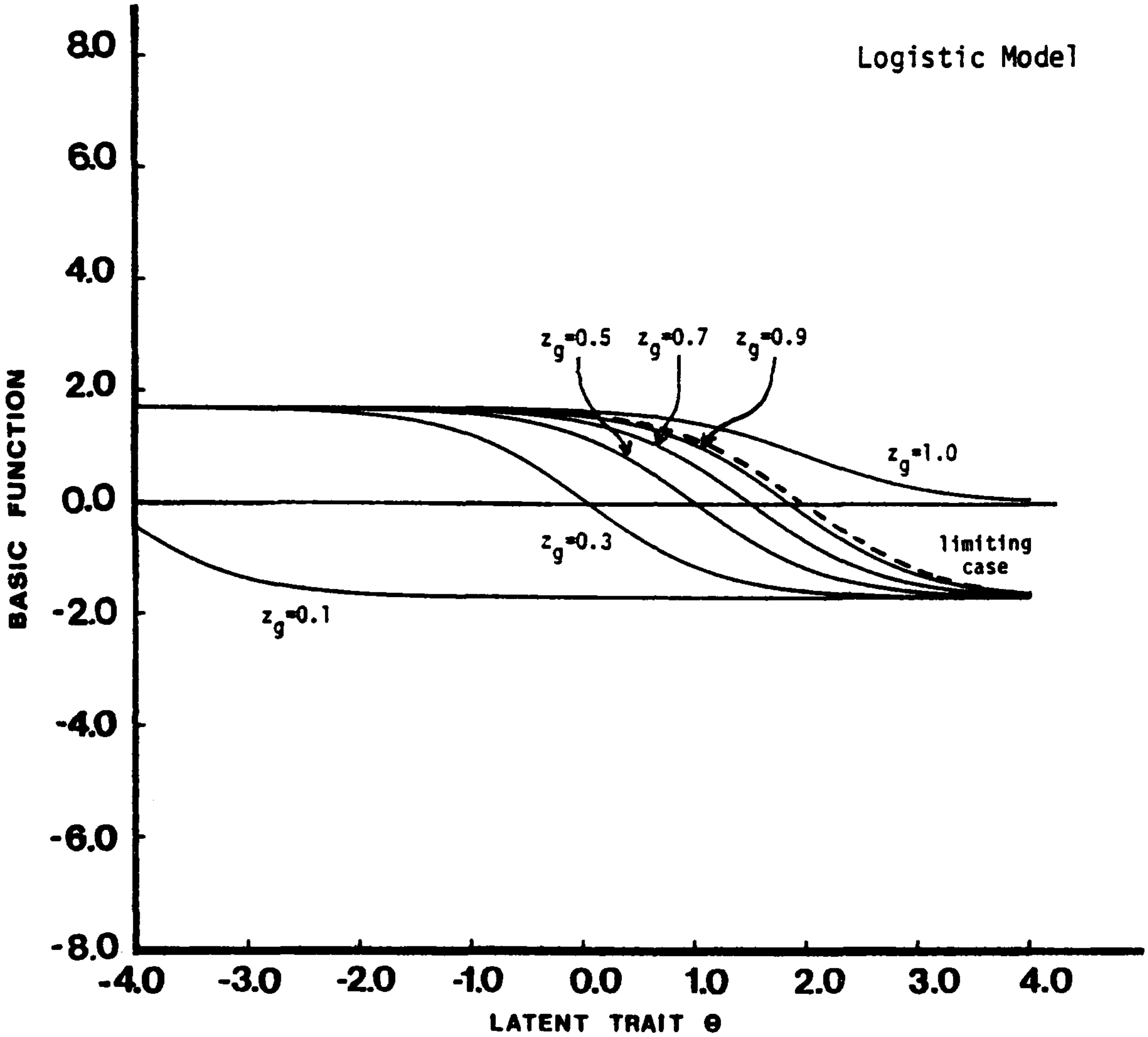

FIGURE 5-8 (Continued)

APPENDIX I (Continued)

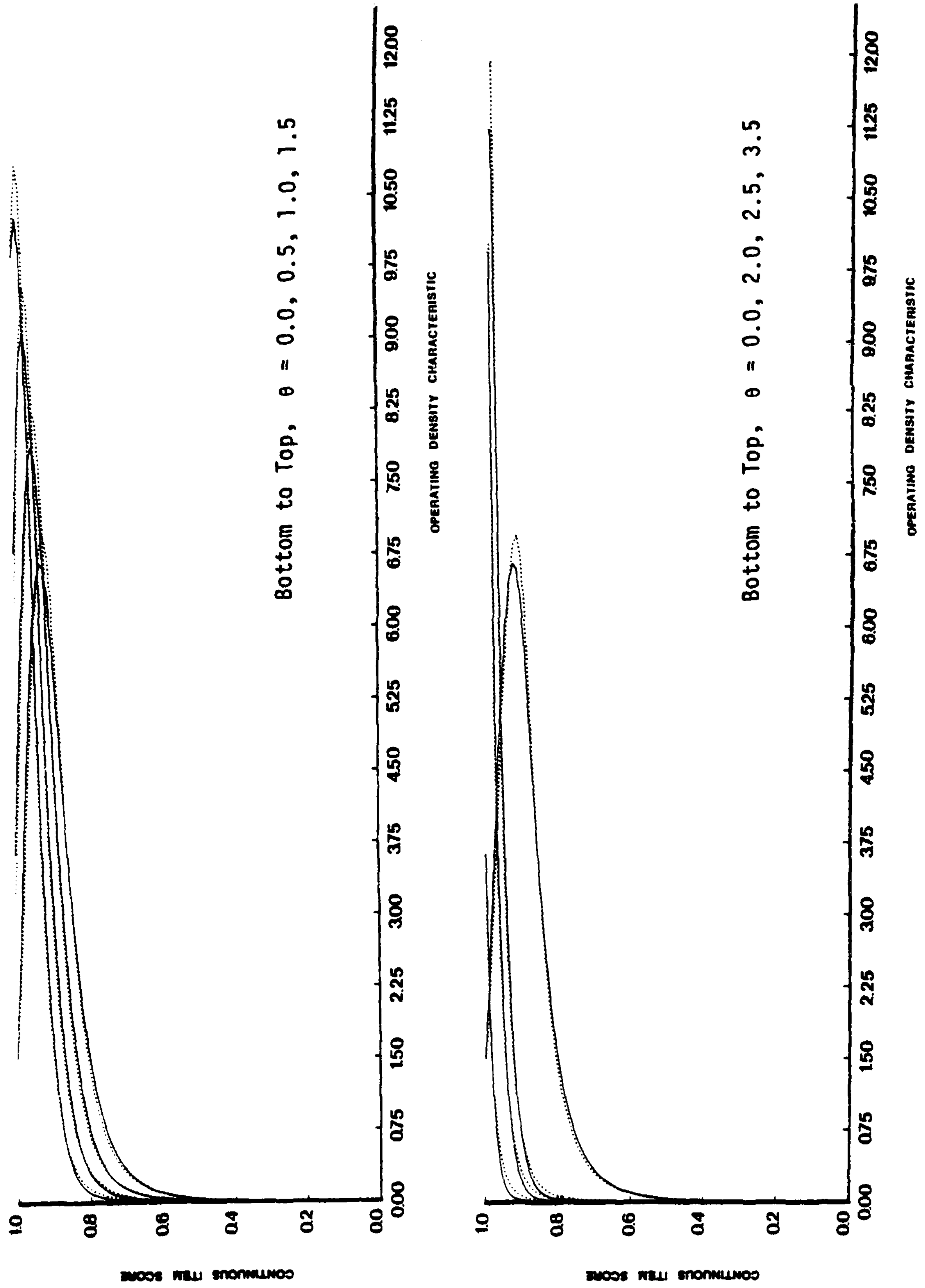

FIGURE A-1-4 (Continued)

Normal Ogive Model (solid line) and Logistic Model (dotted line)

APPENDIX I (Continued)

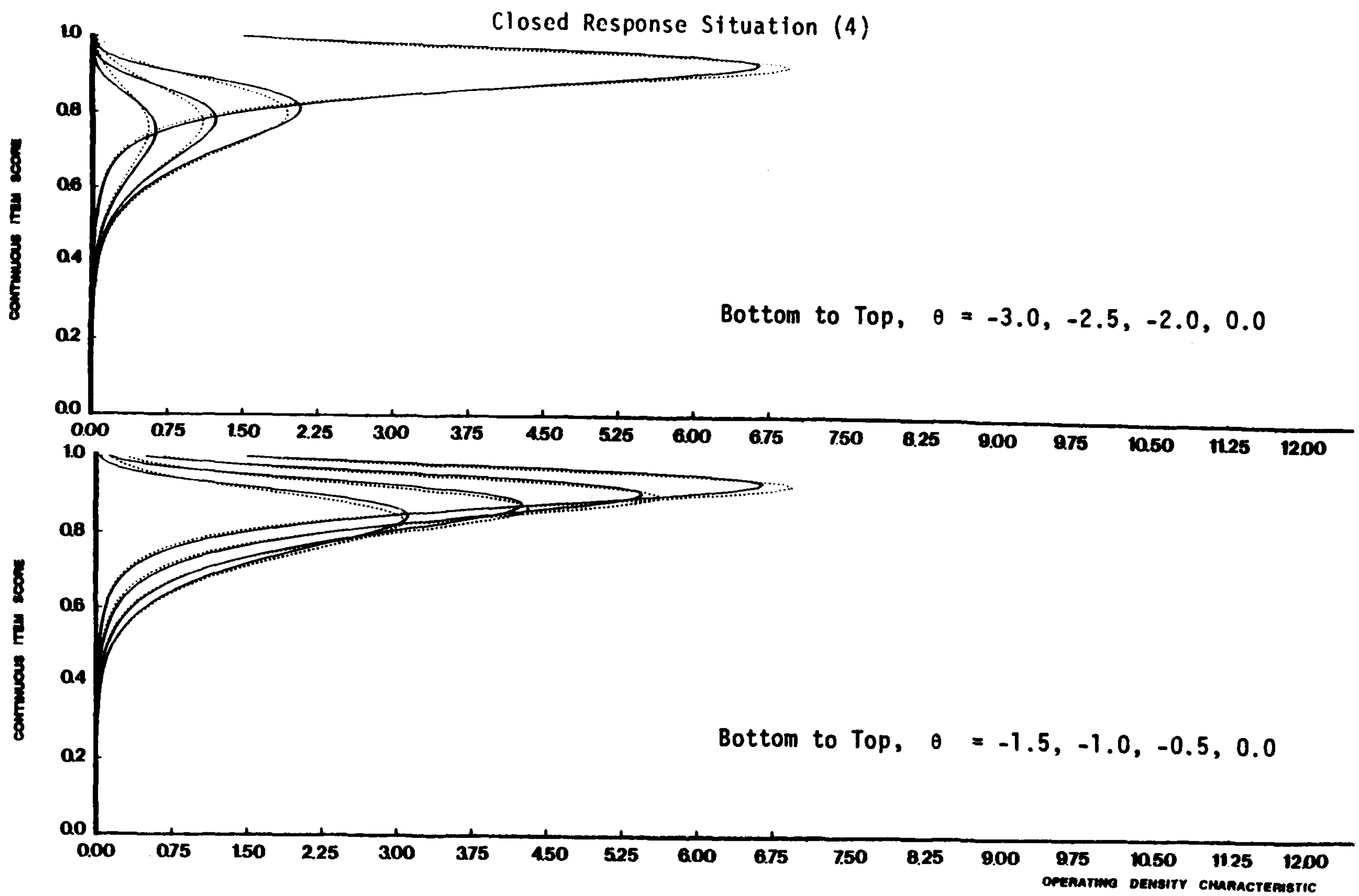

FIGURE A-1-4

Operating Density Characteristic, $H_{z_g}(\theta)$, for Each of the Thirteen Different Values of θ in the Closed Response Situation, in the Normal Ogive Model (solid line) and in the Logistic Model (dotted line), where $a_g = 1.0$ and $D = 1.7$. b_{z_g} is given as a function of Z_g by:

$$b_{z_g} = -2.0 + 4.0\, z_g^{7} .$$

APPENDIX I (Continued)

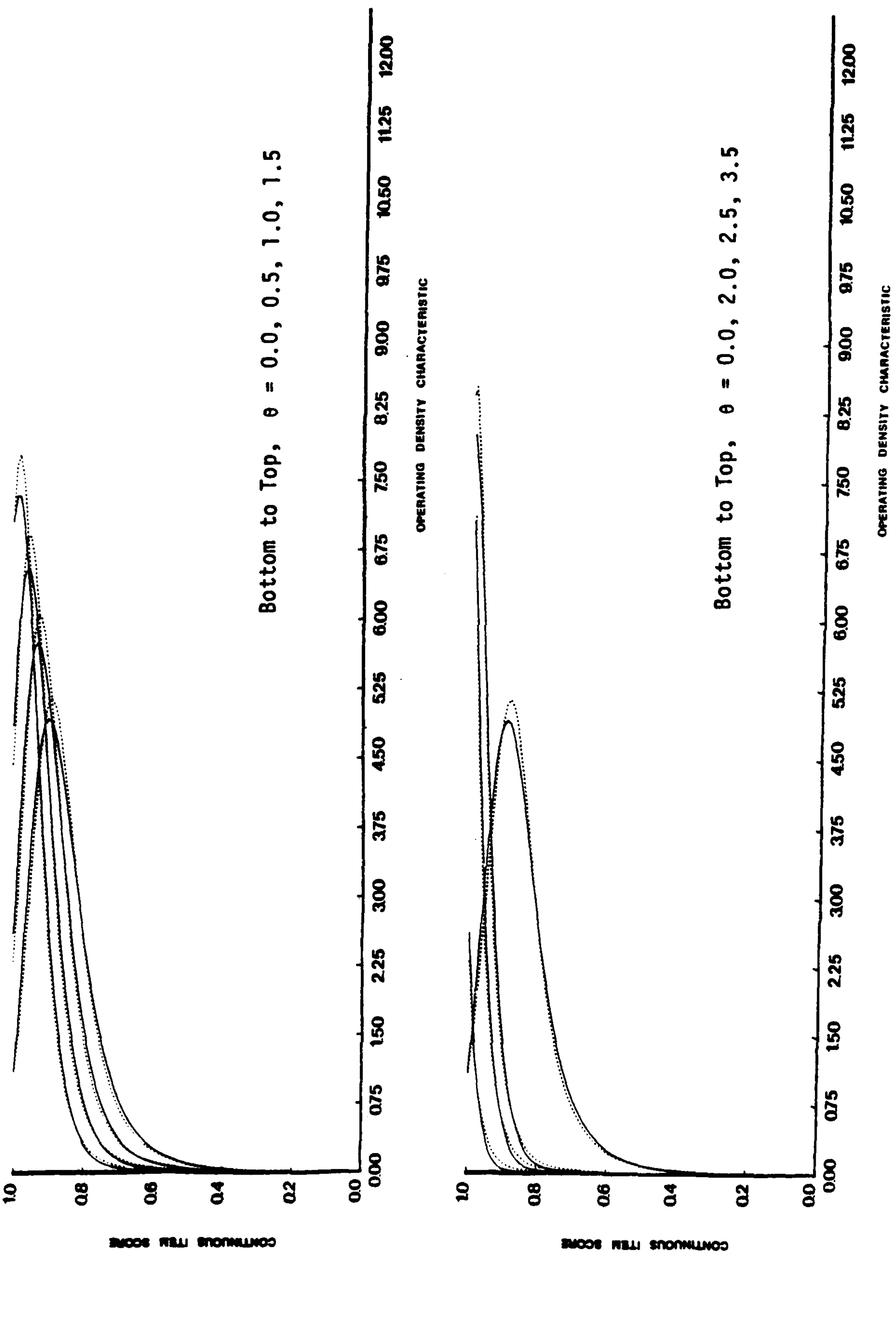

FIGURE A-1-3 (Continued)

Normal Ogive Model (solid line) and Logistic Model (dotted line)

APPENDIX I (Continued)

Closed Response Situation (3)

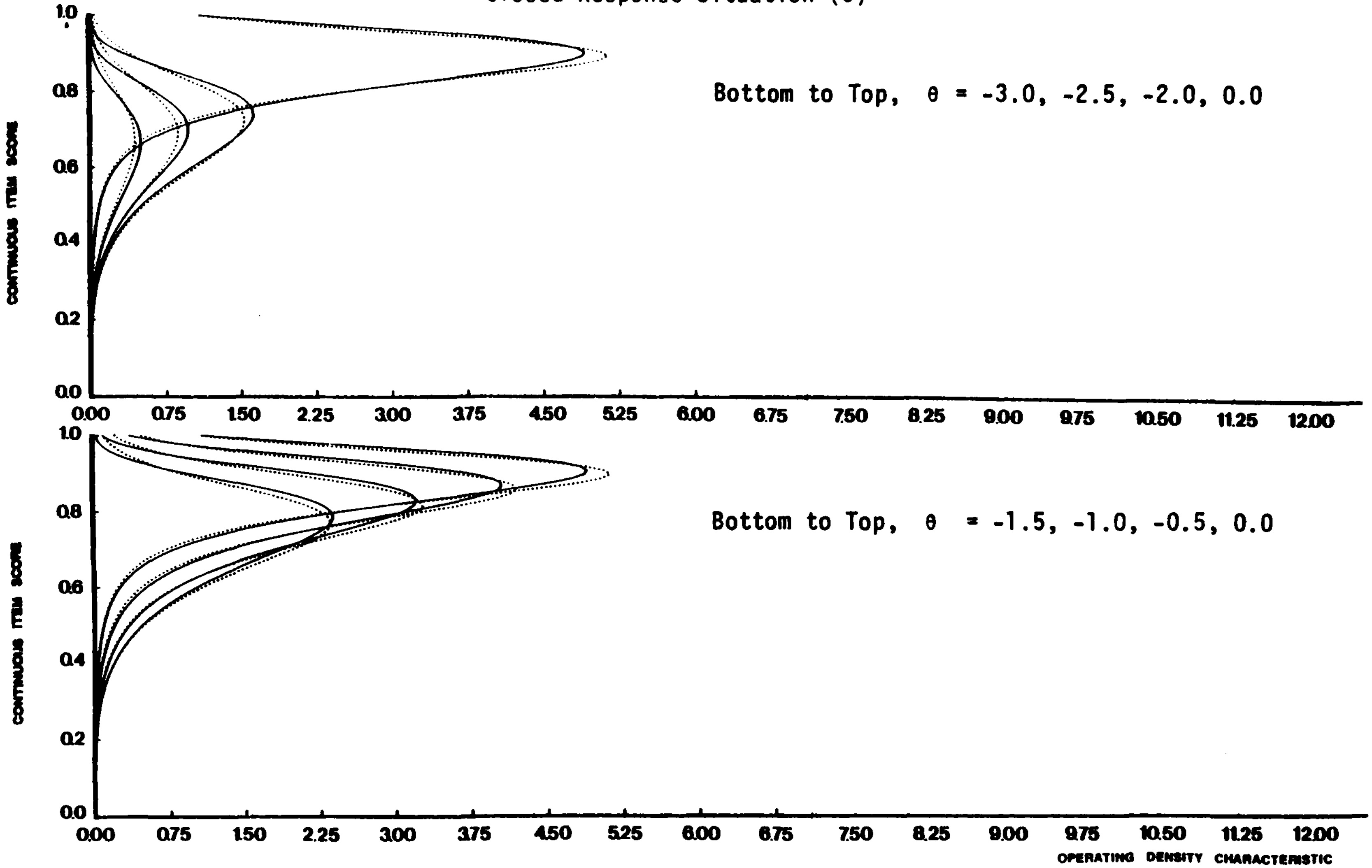

FIGURE A-1-3

Operating Density Characteristic, $H_{z_g}(\theta)$, for Each of the Thirteen Different Values of θ in the Closed Response Situation, in the Normal Ogive Model (solid line) and in the Logistic Model (dotted line), where $a_g = 1.0$ and $D = 1.7$. b_{z_g} is given as a function of Z_g by:

$$b_{z_g} = -2.0 + 4.0\, z_g^{\,5} .$$

APPENDIX I (Continued)

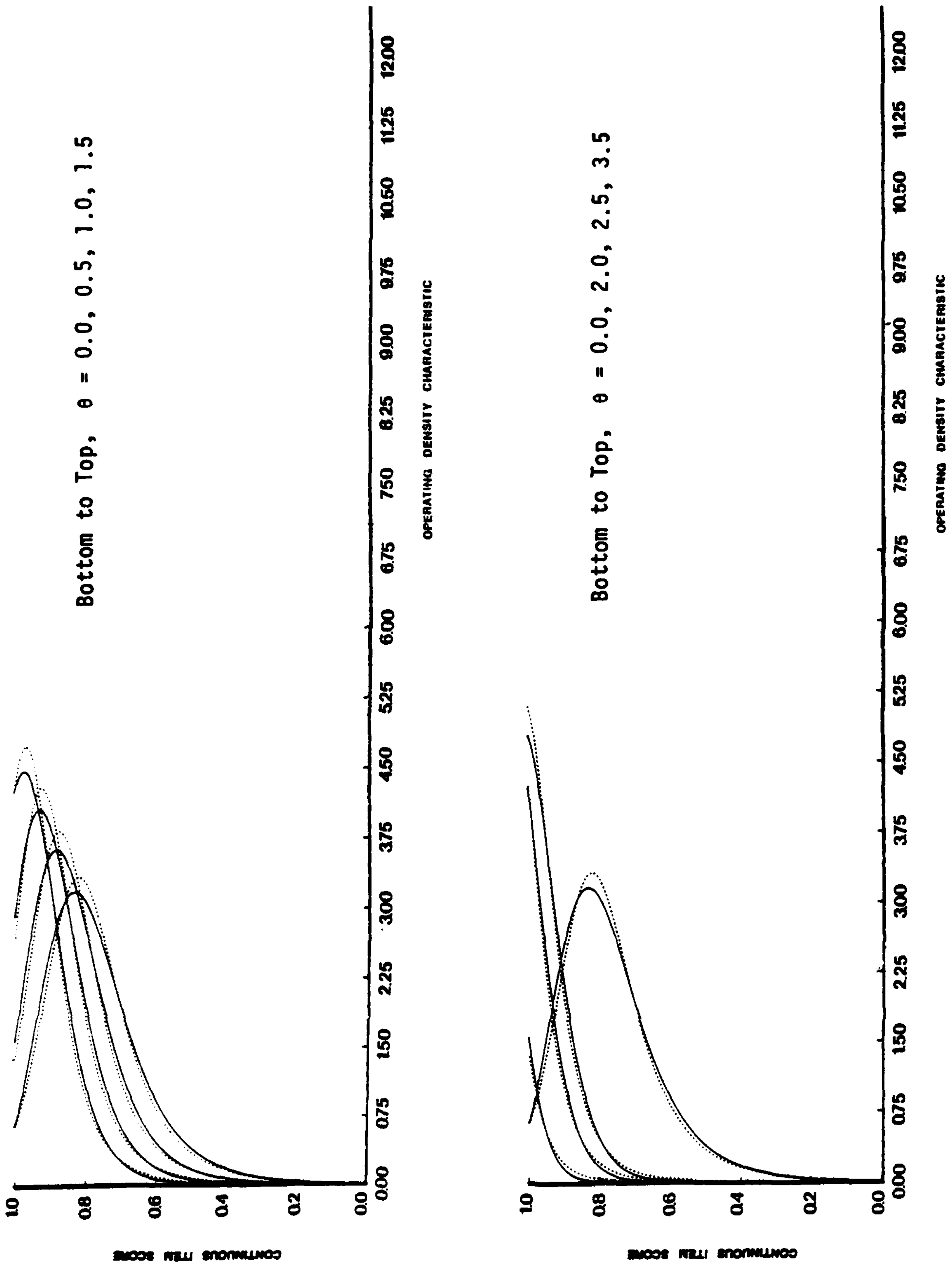

FIGURE A-1-2 (Continued)

Normal Ogive Model (solid line) and Logistic Model (dotted line)

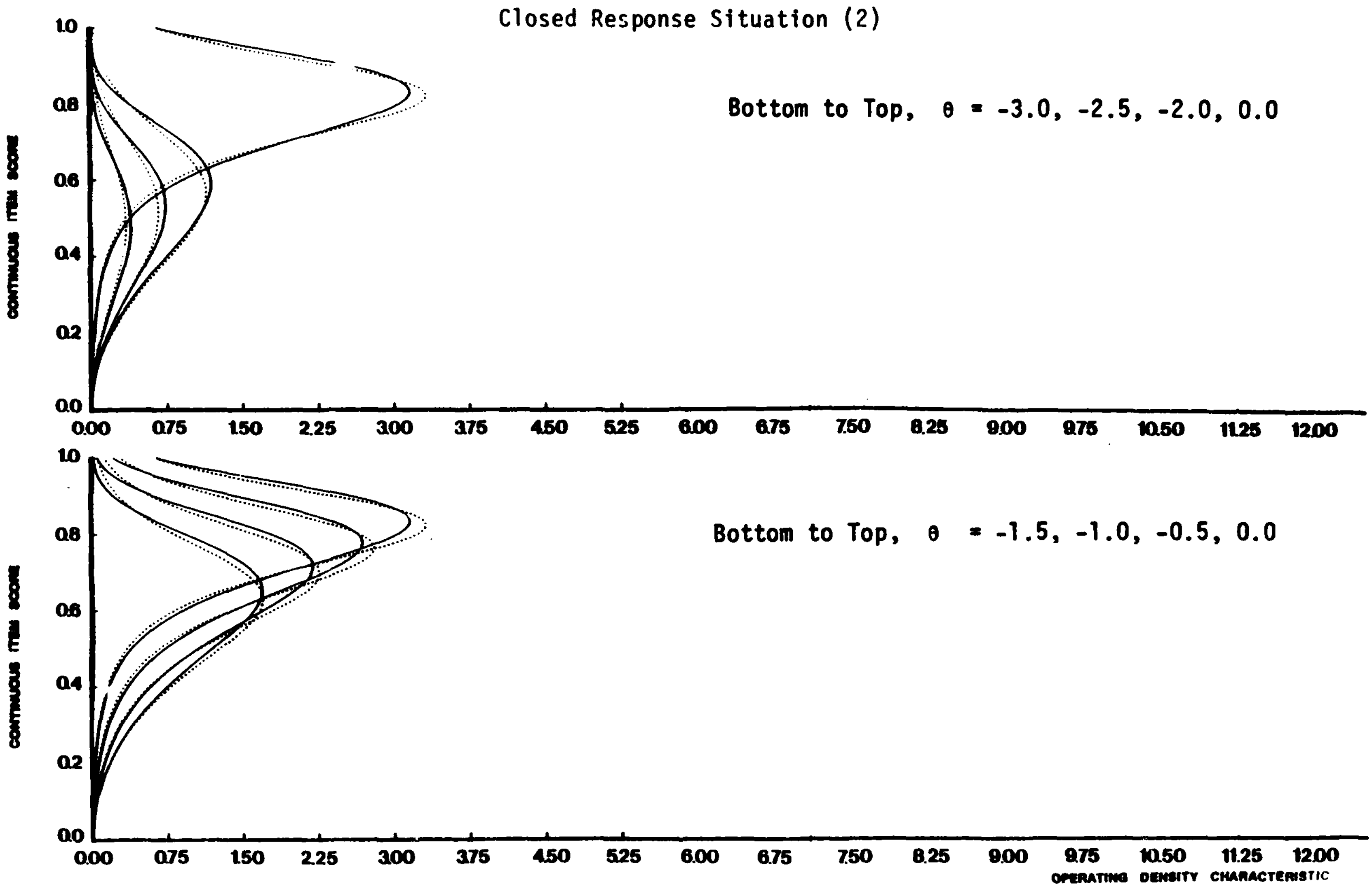

FIGURE A-1-2

Operating Density Characteristic, $H_{z_g}(\theta)$, for Each of the Thirteen Different Values of θ in the Closed Response Situation, in the Normal Ogive Model (solid line) and in the Logistic Model (dotted line), where $a_g = 1.0$ and $D = 1.7$. b_{z_g} is given as a function of Z_g by:

$$b_{z_g} = -2.0 + 4.0 z_g^3 .$$

APPENDIX I (Continued)

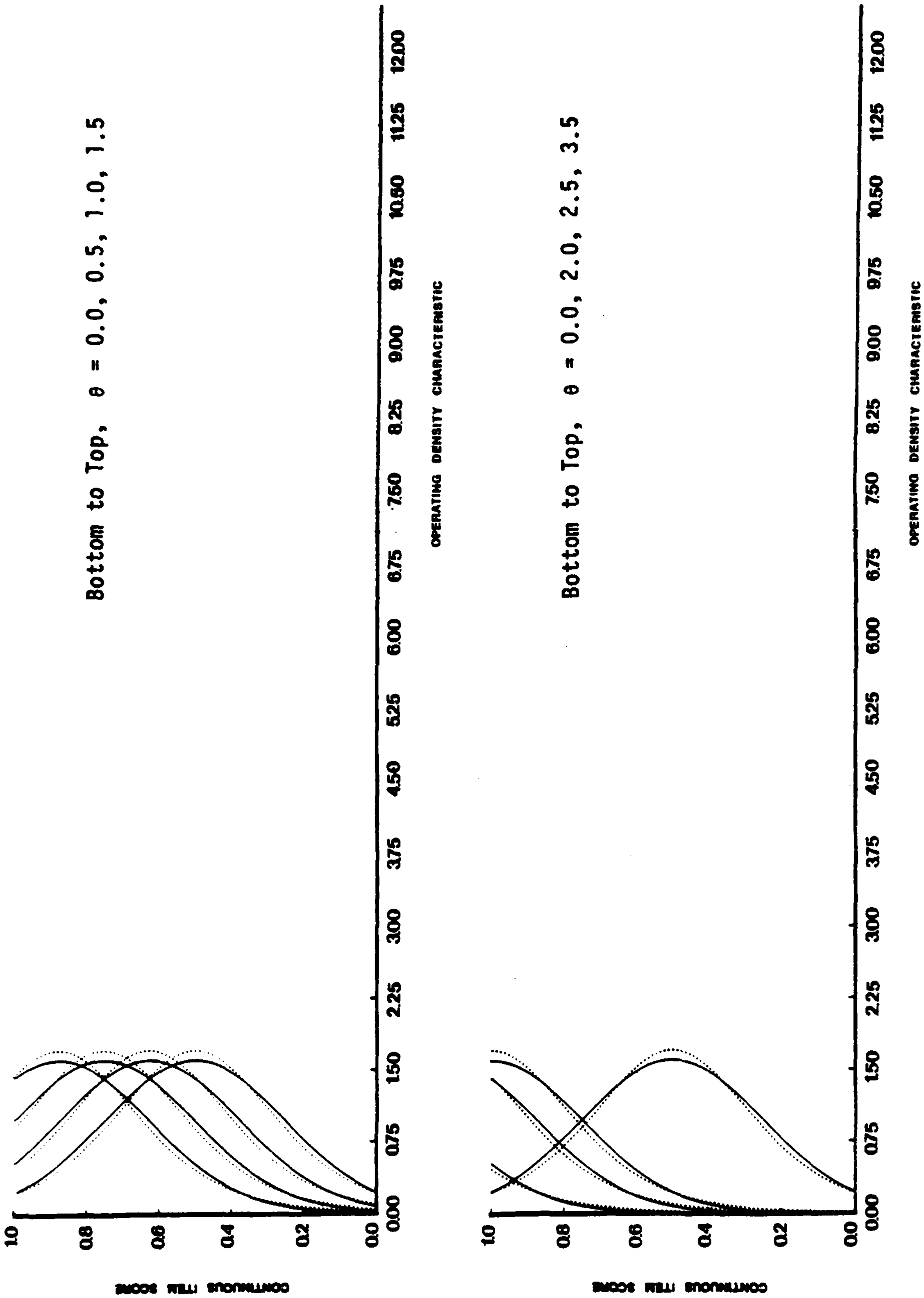

FIGURE A-1-1 (Continued)

Normal Ogive Model (solid line) and Logistic Model (dotted line)

APPENDIX I

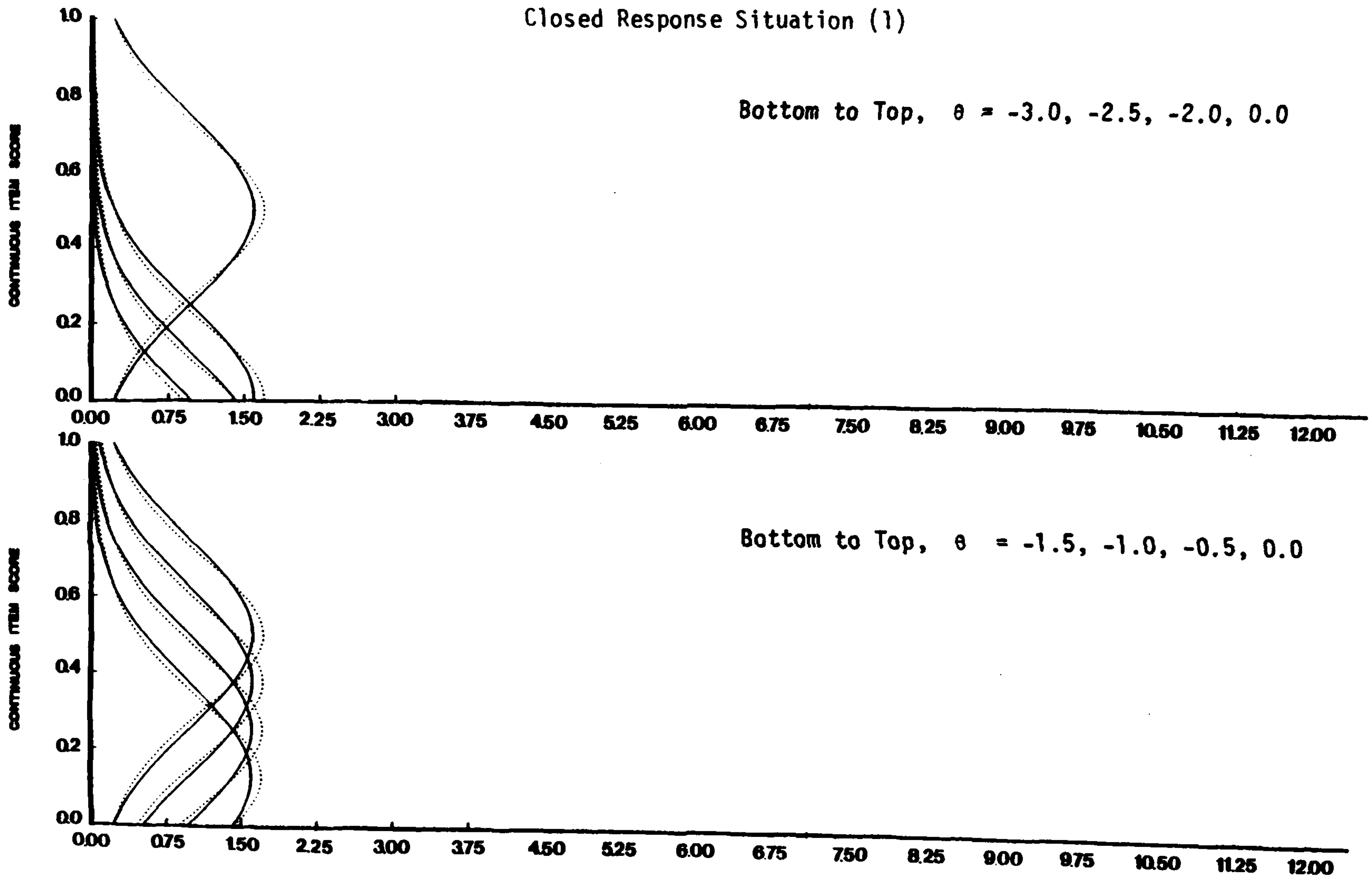

FIGURE A-1-1

Operating Density Characteristic, $H_{z_g}(\theta)$, for Each of the Thirteen Different Values of θ in the Closed Response Situation, in the Normal Ogive Model (solid line) and in the Logistic Model (dotted line), where $a_g = 1.0$ and $D = 1.7$. b_{z_g} is given as a function of z_g by:

$$b_{z_g} = -2.0 + 4.0\, z_g$$

APPENDICES

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[12] Samejima, F. Homogeneous case of the continuous response level. Psychometrika, 1973, 38, 203-219.

[13] Samejima, F. Normal ogive model on the continuous response level in the multidimensional latent space. Psychometrika, 1974, 39, 111-121.

Two specific models, i.e., the normal ogive model and the logistic model, have been proposed and investigated in each of the three response situations. It is left to the researcher to decide which model to choose for his or her own research. It should be emphasized, however, that the failure in the selection of an appropriate model will necessarily lead to the failure of research. The example of the response latency in cognitive psychology given in earlier chapters may be helpful in such a decision making.

The models proposed here will further be expanded and generalized to include situations in which any finite or enumerable set of z_g between 0 and 1 deals with the discrete part of the conditional distribution of the item response z_g, given the latent trait θ. When necessities come, the rationale presented in the present paper is ready to be expanded and generalized further. In such a situation the general model in the closed response situation will become a special case of the even larger framework.

and the item information function $I_g(\theta)$ by solid lines and a dotted line, respectively, in each of the normal ogive and the logistic models, with the same parameters, scaling factor, difficulty parameter function and six fixed values of z_g as we used in Figures 5-7 and 5-8. As was the case with the closed response and the closed/open response situations, in the normal ogive model, the item response information function for each and every value of z_g in the interval $(0,1)$ has the identical horizontal line shown in the upper graph of Figure 5-9, and, therefore, this single curve serves for all the five values of z_g, i.e., 0.1, 0.3, 0.5, 0.7 and 0.9. It should also be noted that this graph is valid across different difficulty parameter functions. The item response information function in the limiting case where z_g tends to unity is overlapping the solid horizontal curve in the normal ogive model, and shown by a dashed curve in the logistic model, in Figure 5-9. In each model, the item response information function for $z_g = 1.0$ is less than the one in the limiting case, where z_g tends to unity, for the entire range of θ, as was the case with the closed/open response situation with the replacement of unity by zero. Similar sets of eight curves in the logistic model, obtained by changing k to 3, 5, 7 and 9 in the difficulty parameter function given by (5.24), respectively, are shown in Figure A-12 of Appendix XII.

VI Discussion and Conclusions

A generalization of the model for the open response situation in the homogeneous case of the continuous response level has been conducted to create a new general model for the closed response situation, which also includes the closed/open response situation and the open/closed response situation, as well as the open response situation itself, as special cases.

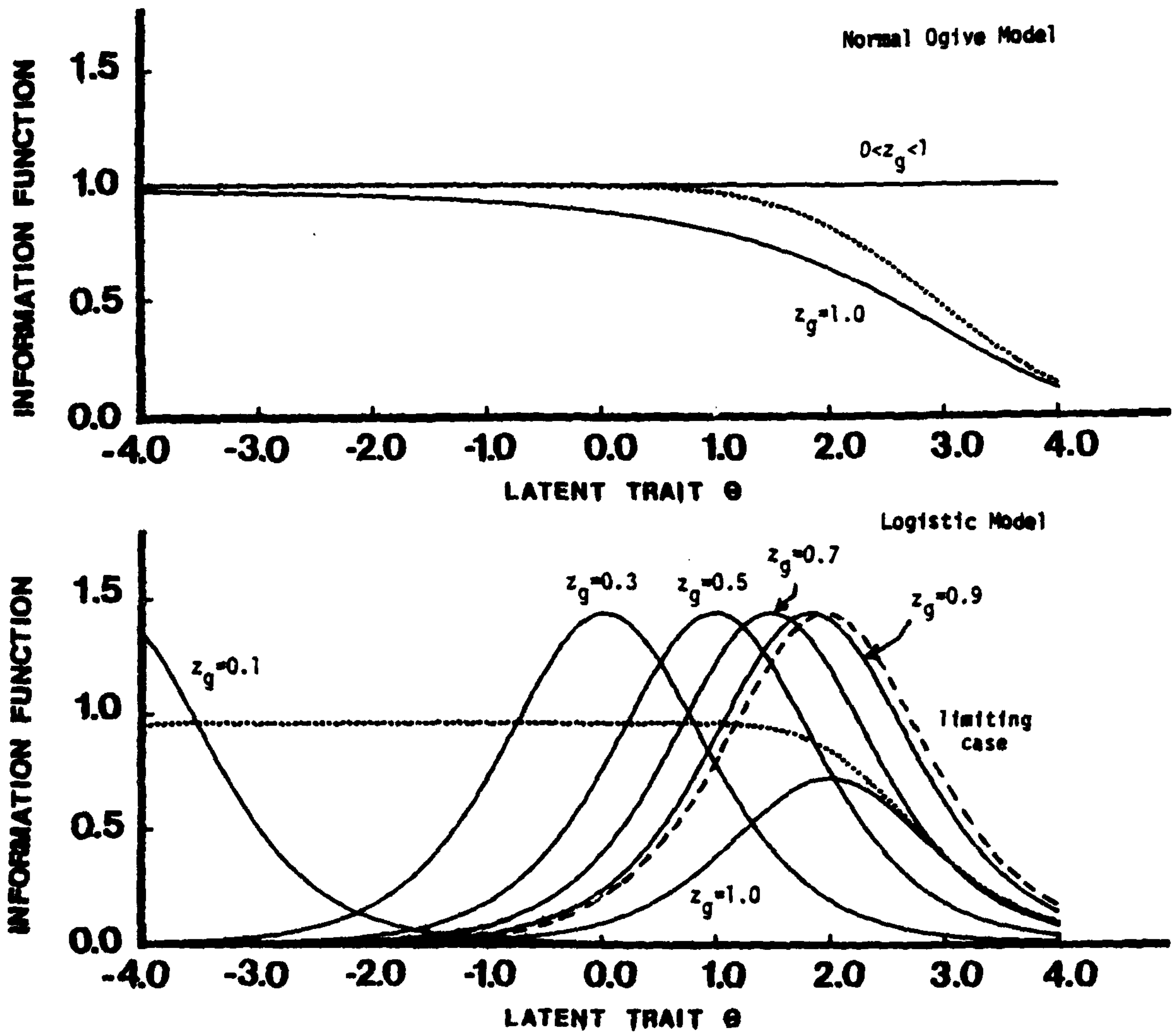


FIGURE 5-9

Item Response Information Functions, $I_{z_g}(\theta)$, (Solid Line) and Item Information Function, $I_g(\theta)$, (Dotted Line) in the Normal Ogive and the Logistic Models, with $a_g = 1.0$, $b_1 = 2.0$ and $D = 1.7$. In the Normal Ogive Model, the Horizontal Line Indicates Common $I_{z_g}(\theta)$ for All Item Scores, $0 < z_g < 1$, While in the Logistic Model the Five Curves Identical in Shape Indicate $I_{z_g}(\theta)$ for $z_g = 0.1, 0.3, 0.5, 0.7, 0.9$, When the Functional Relationship between z_g and b_{z_g} Is Given by $b_{z_g} = b_1 + \tan[(-\pi/2)(1-z_g)]$, with the Dashed Curve as the One in the Limiting Situation Where z_g Tends to Unity. Open/Closed Response Situation.

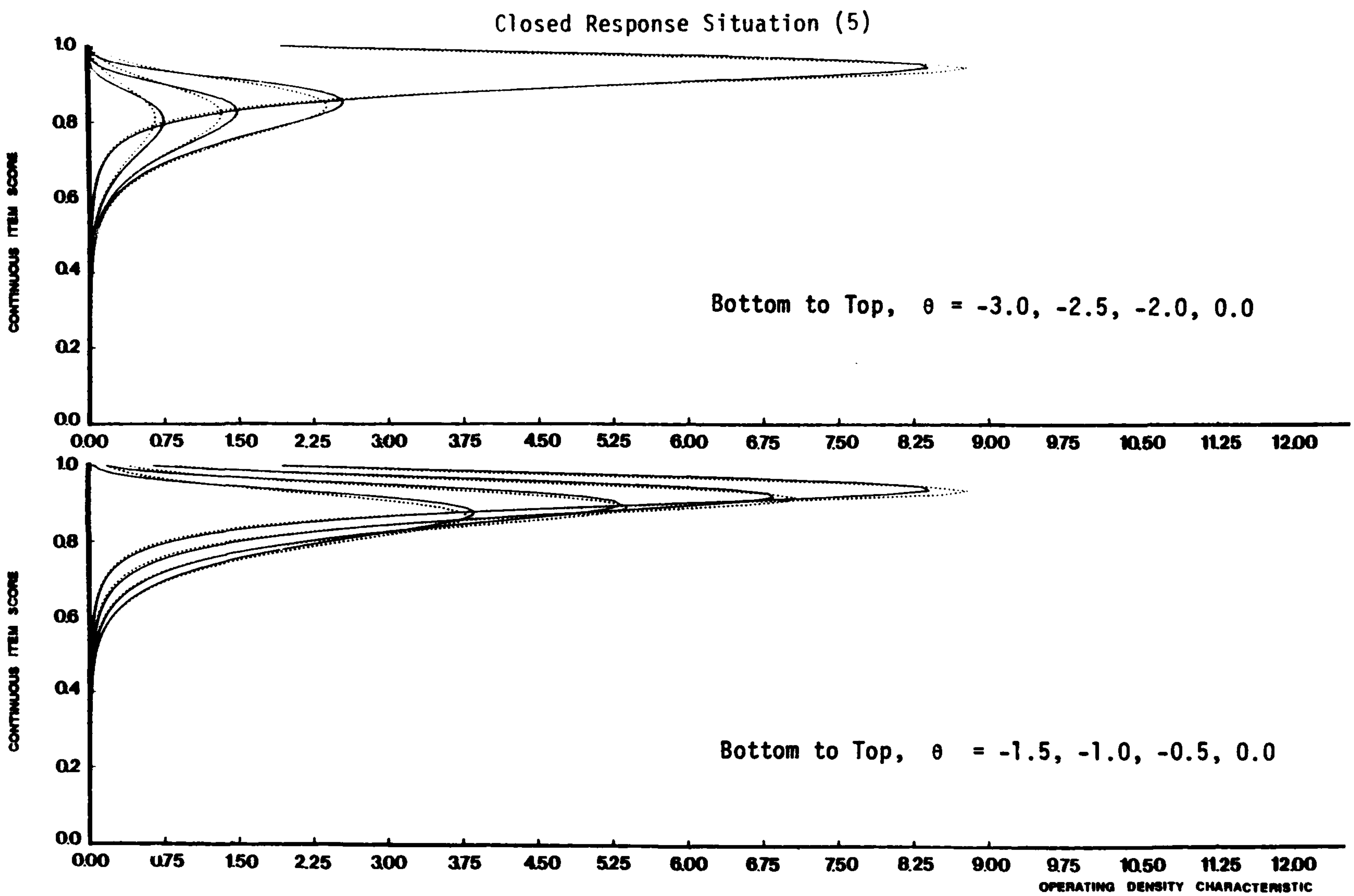


FIGURE A-1-5

Operating Density Characteristic, $H_{z_g}(\theta)$, for Each of the Thirteen Different Values of θ in the Closed Response Situation, in the Normal Ogive Model (solid line) and in the Logistic Model (dotted line), where $a_g = 1.0$ and $D = 1.7$. b_{z_g} is given as a function of Z_g by:

$$b_{z_g} = -2.0 + 4.0\, z_g^{\,9} .$$

APPENDIX I (Continued)

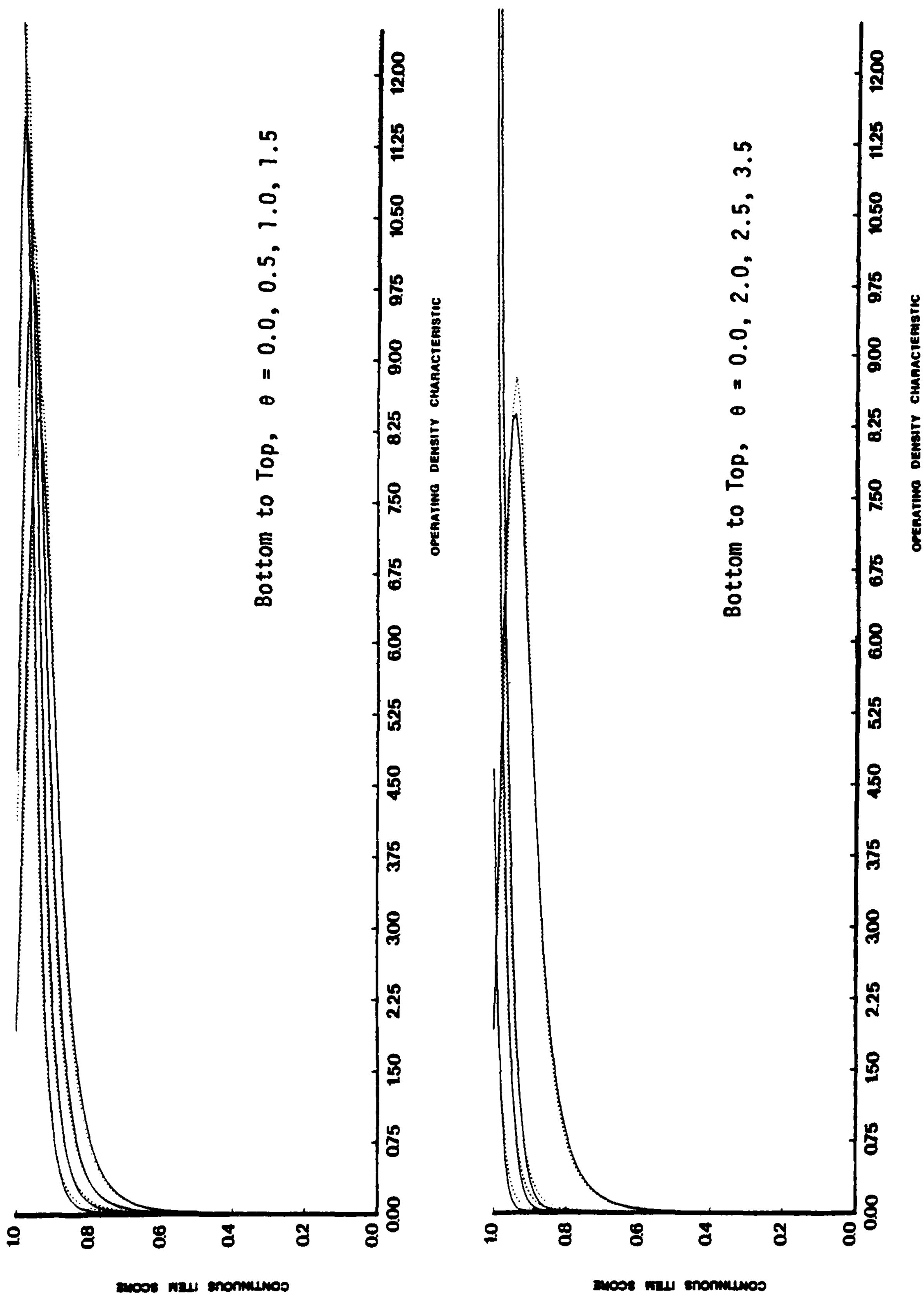


FIGURE A-1-5 (Continued)

Normal Ogive Model (solid line) and Logistic Model (dotted line)

APPENDIX II

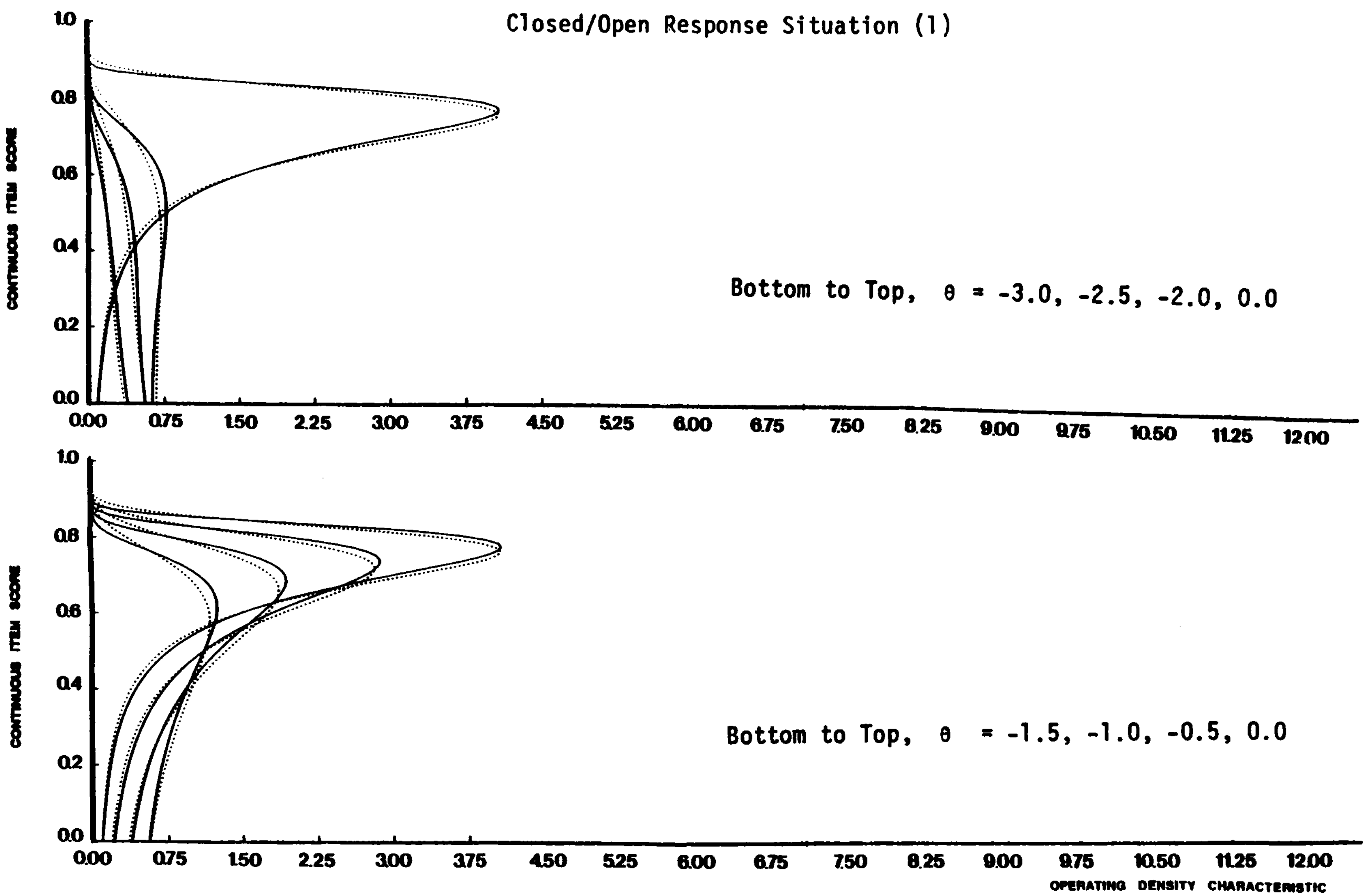
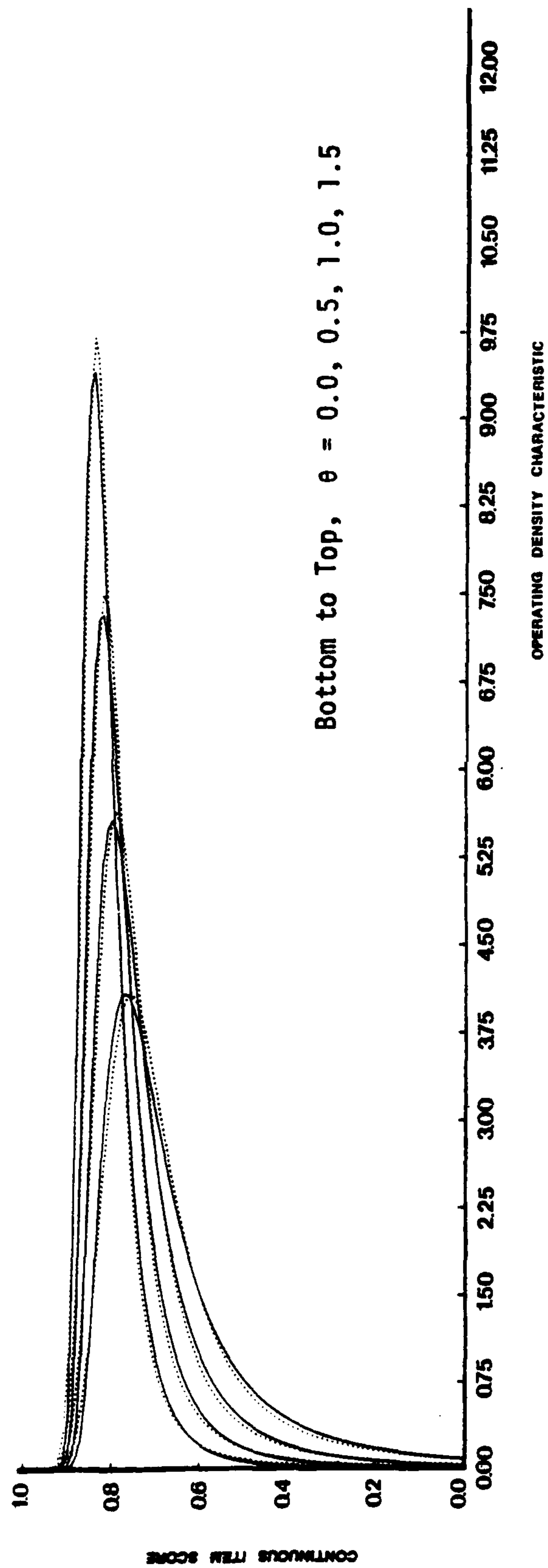


FIGURE A-2-1

Operating Density Characteristic, $H_{z_g}(\theta)$, for Each of the Thirteen Different Values of θ in the Closed/Open Response Situation, in the Normal Ogive Model (solid line) and in the Logistic Model (dotted line), where $a_g = 1.0$ and $D = 1.7$. b_{z_g} is given as a function of Z_g by:

$$b_{z_g} = -2.0 + \tan [(\pi/2)(1.0\ z_g)] .$$

APPENDIX II (Continued)



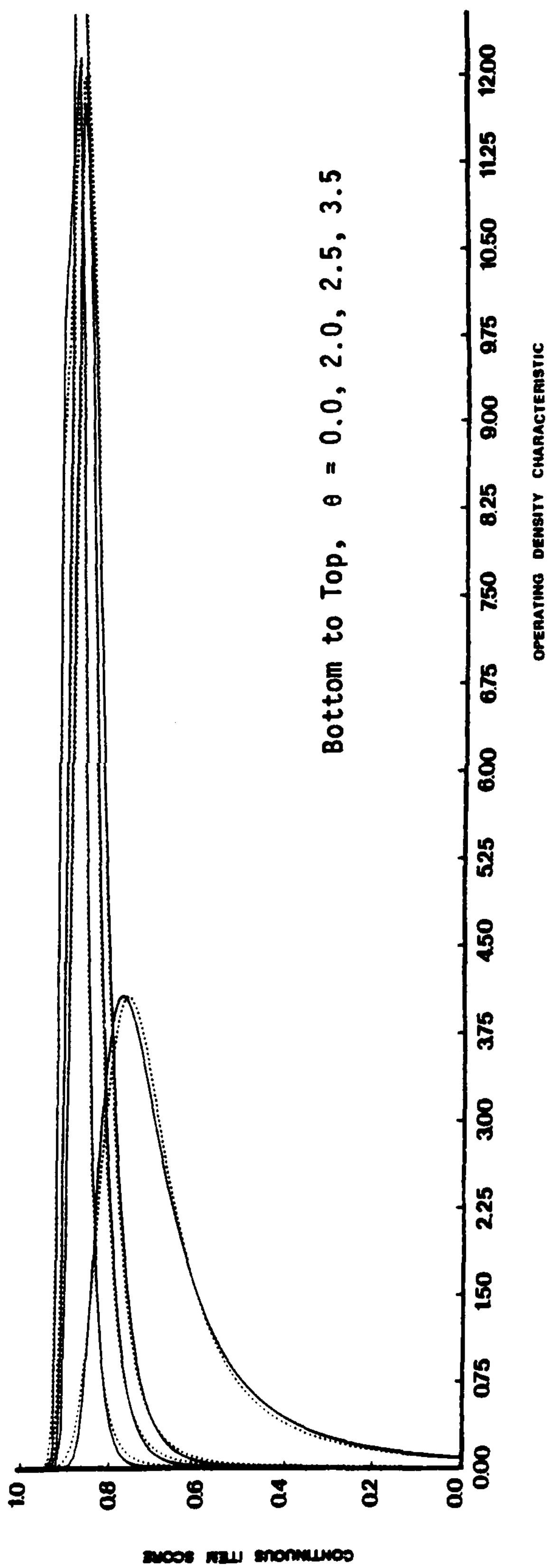


FIGURE A-2-1 (Continued)

Normal Ogive Model (solid line) and Logistic Model (dotted line)

APPENDIX II (Continued)

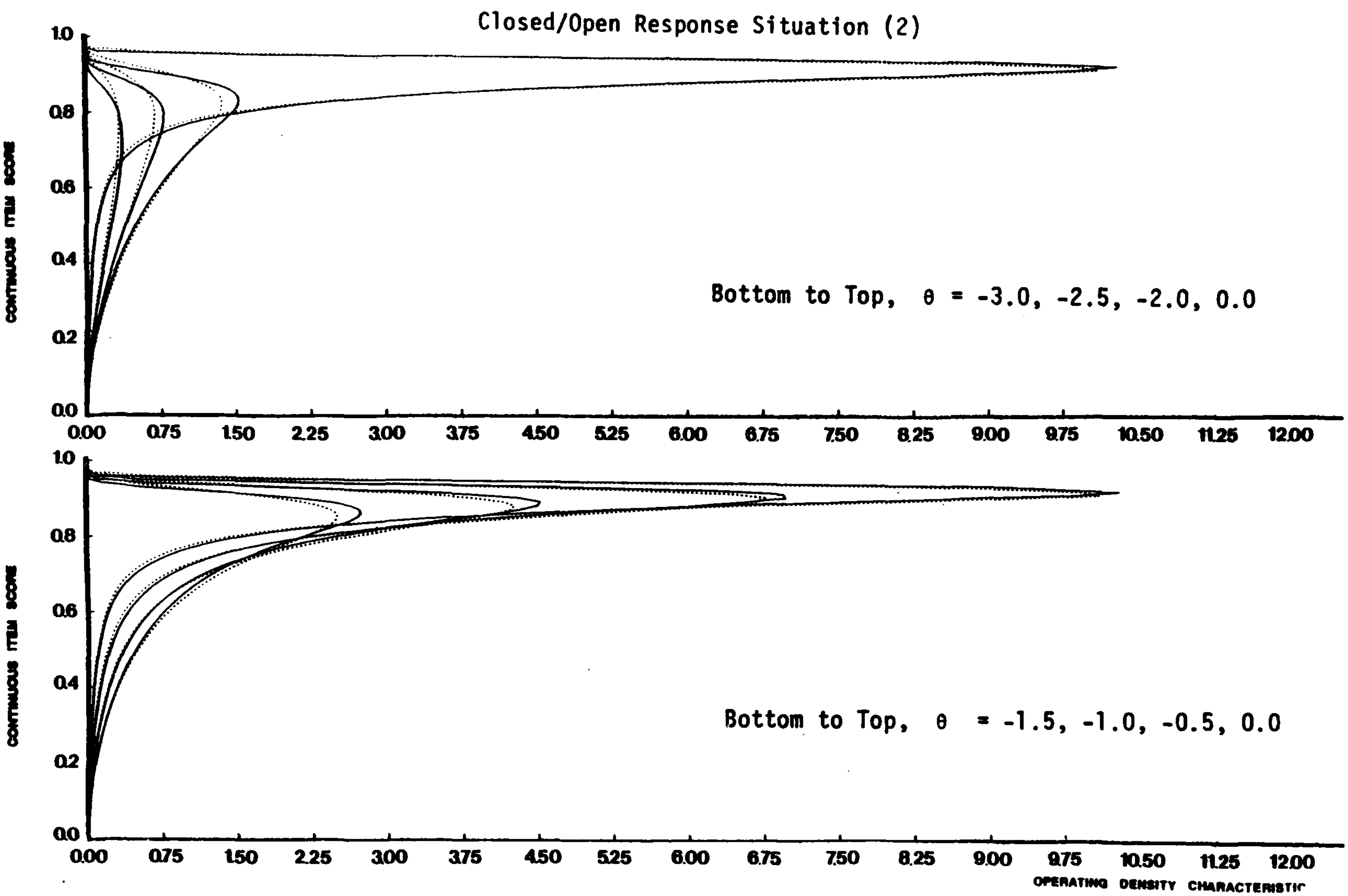


FIGURE A-2-2

Operating Density Characteristic, $H_{z_g}(\theta)$, for Each of the Thirteen Different Values of θ in the Closed/Open Response Situation, in the Normal Ogive Model (solid line) and in the Logistic Model (dotted line), where $a_g = 1.0$ and $D = 1.7$. b_{z_g} is given as a function of Z_g by:

$$b_{z_g} = -2.0 + \tan [(\pi/2)(1.0\ z_g^3)] .$$

APPENDIX II (Continued)

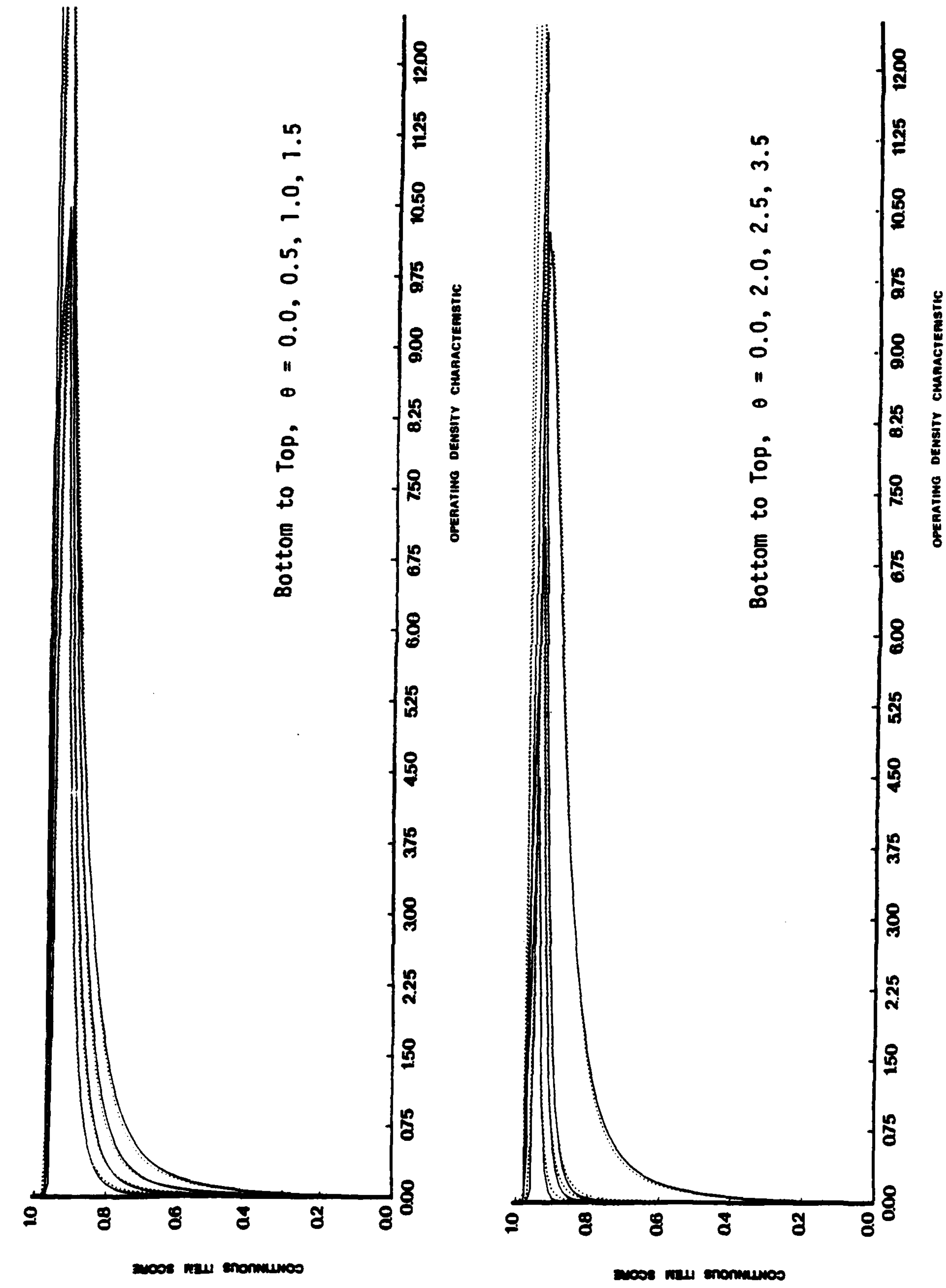


FIGURE A-2-2 (Continued)

Normal Ogive Model (solid line) and Logistic Model (dotted line)

APPENDIX II (Continued)

Closed/Open Response Situation (3)

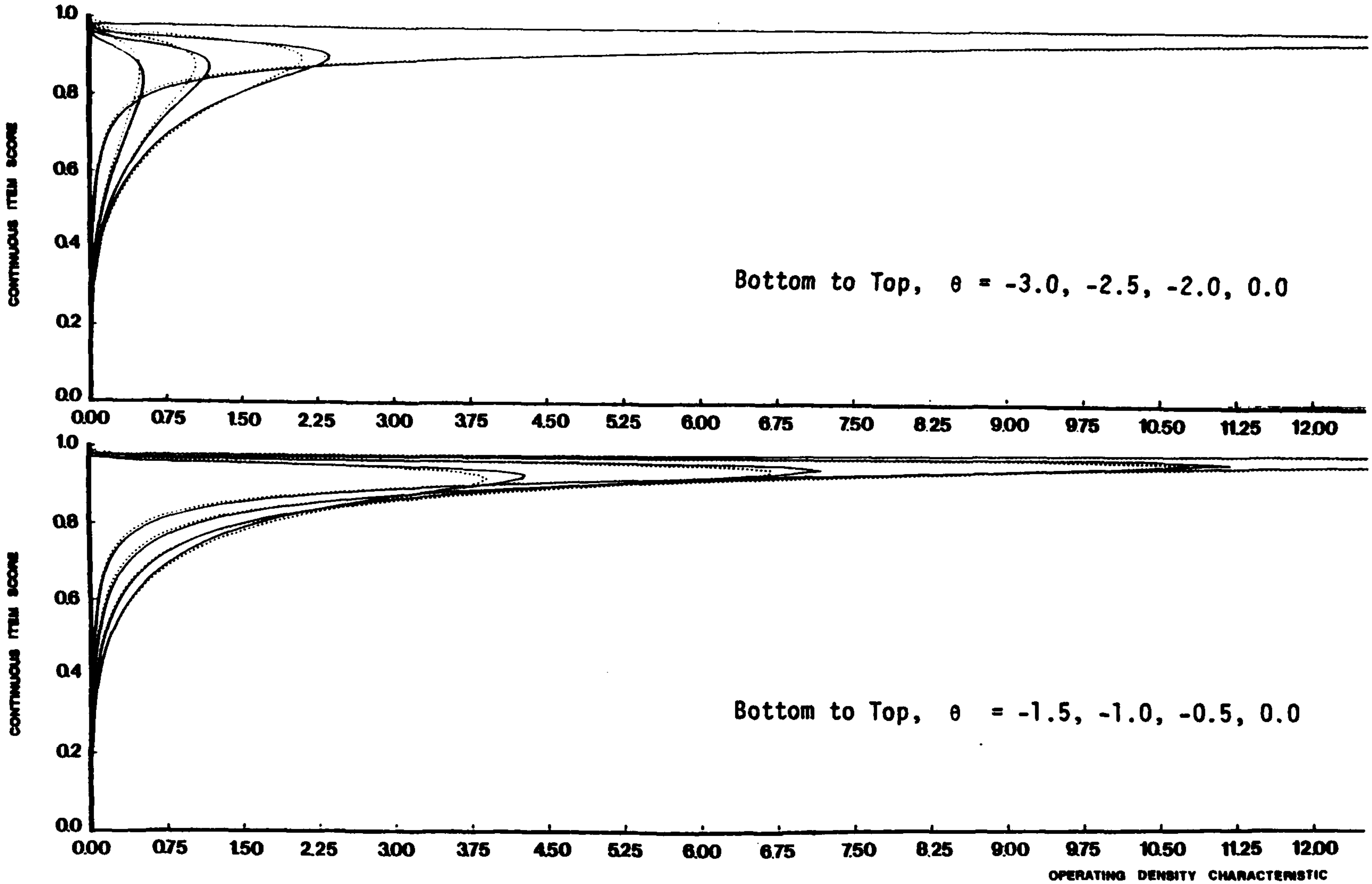


FIGURE A-2-3

Operating Density Characteristic, $H_{z_g}(\theta)$, for Each of the Thirteen Different Values of θ in the Closed/Open Response Situation, in the Normal Ogive Model (solid line) and in the Logistic Model (dotted line), where $a_g = 1.0$ and $D = 1.7$. b_{z_g} is given as a function of Z_g by:

$$b_{z_g} = -2.0 + \tan[(\pi/2)(1.0\, z_g^{5})].$$

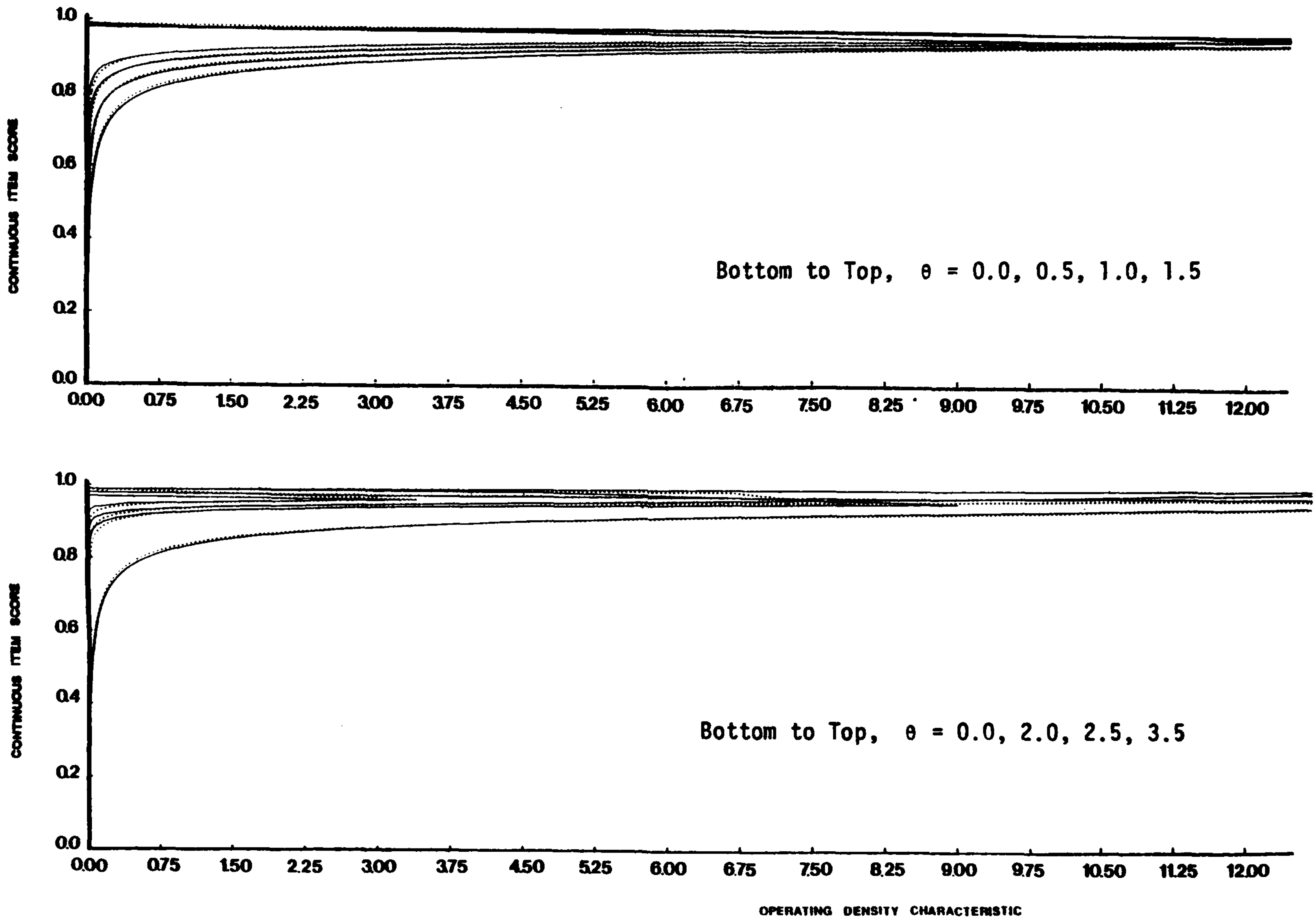


FIGURE A-2-3 (Continued)

Normal Ogive Model (solid line) and Logistic Model (dotted line)

Closed/Open Response Situation (4)

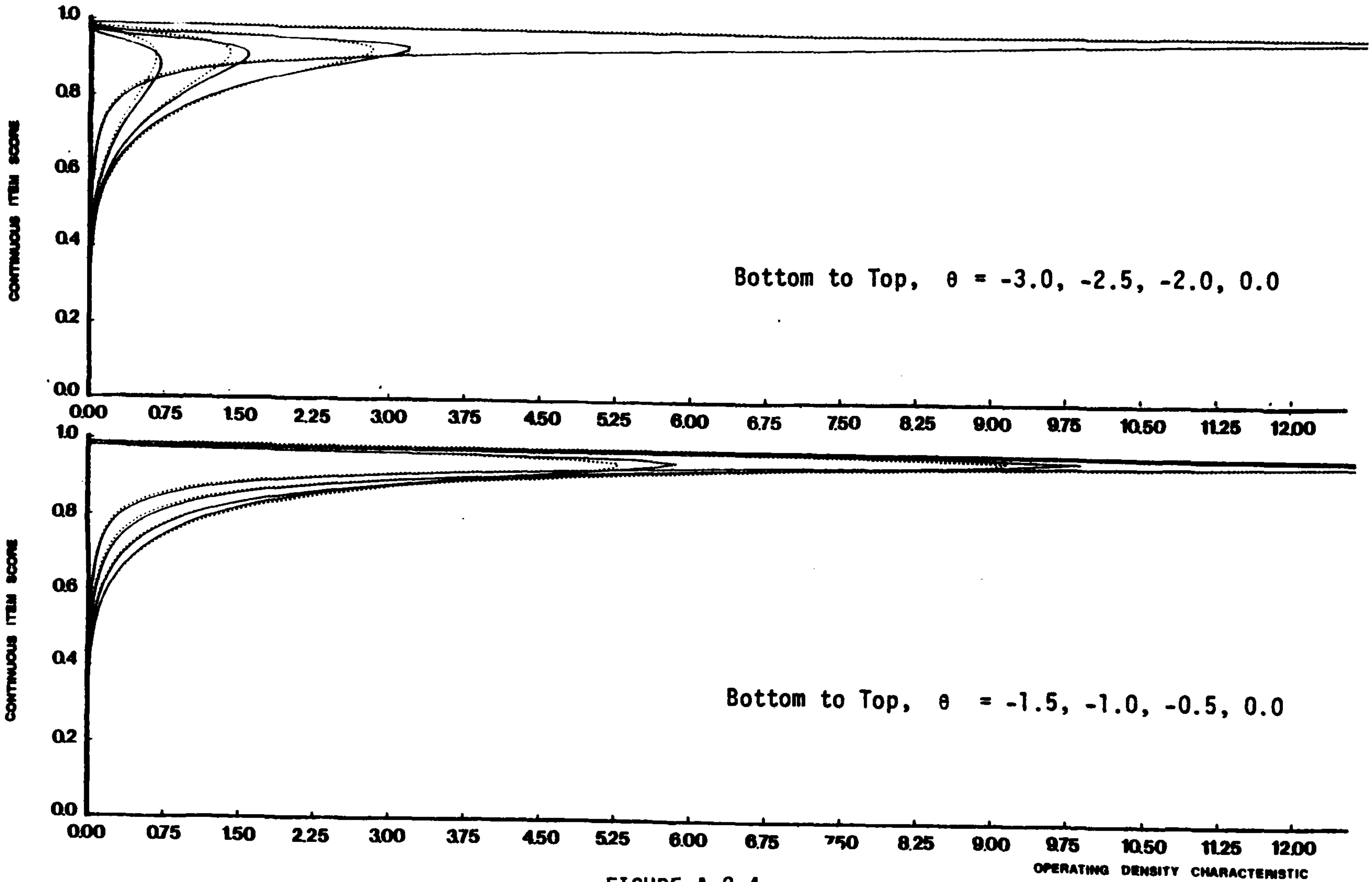


FIGURE A-2-4

Operating Density Characteristic, $H_{z_g}(\theta)$, for Each of the Thirteen Different Values of θ in the Closed/Open Response Situation, in the Normal Ogive Model (solid line) and in the Logistic Model (dotted line), where $a_g = 1.0$ and $D = 1.7$. b_{z_g} is given as a function of Z_g by:

$$b_{z_g} = -2.0 + \tan\left[(\pi/2)(1.0\ z_g^{\,7})\right] .$$

APPENDIX II (Continued)

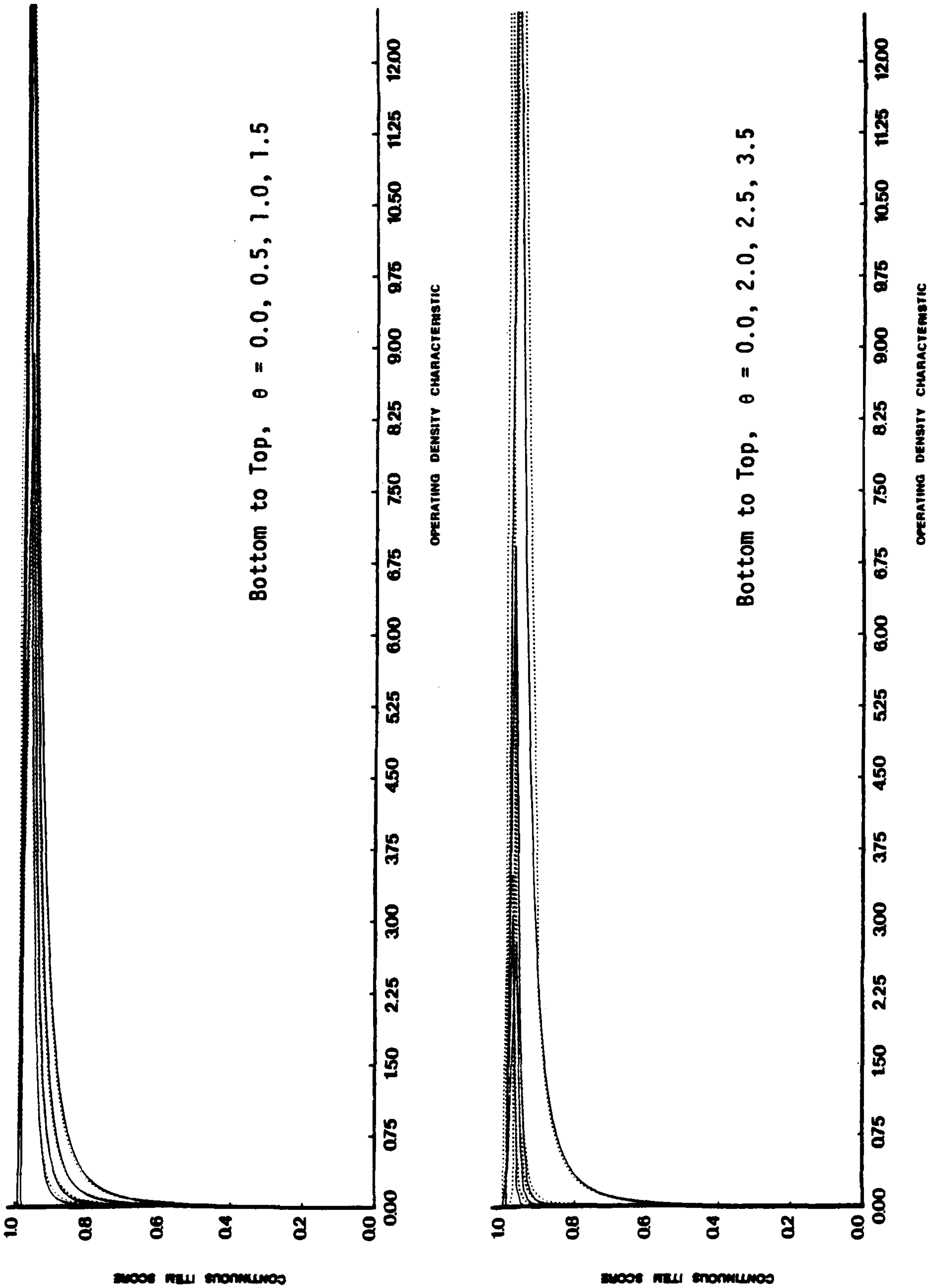


FIGURE A-2-4 (Continued)

Normal Ogive Model (solid line) and Logistic Model (dotted line)

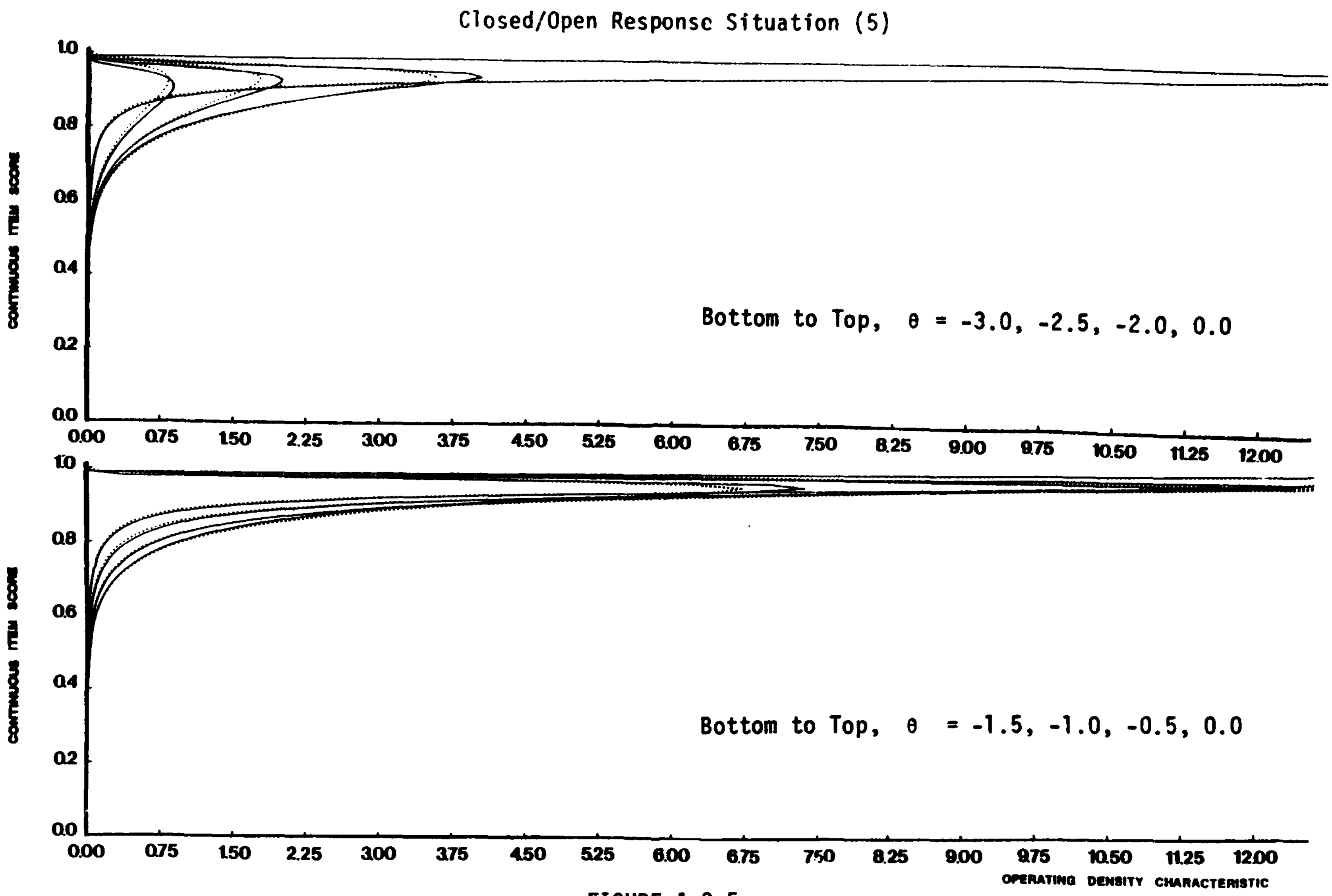


FIGURE A-2-5

Operating Density Characteristic, $H_{z_g}(\theta)$, for Each of the Thirteen Different Values of θ in the Closed/Open Response Situation, in the Normal Ogive Model (solid line) and in the Logistic Model (dotted line), where $a_g = 1.0$ and $D = 1.7$. b_{z_g} is given as a function of Z_g by:

$$b_{z_g} = -2.0 + \tan [(\pi/2)(1.0 z_g{}^9)] .$$

APPENDIX II (Continued)

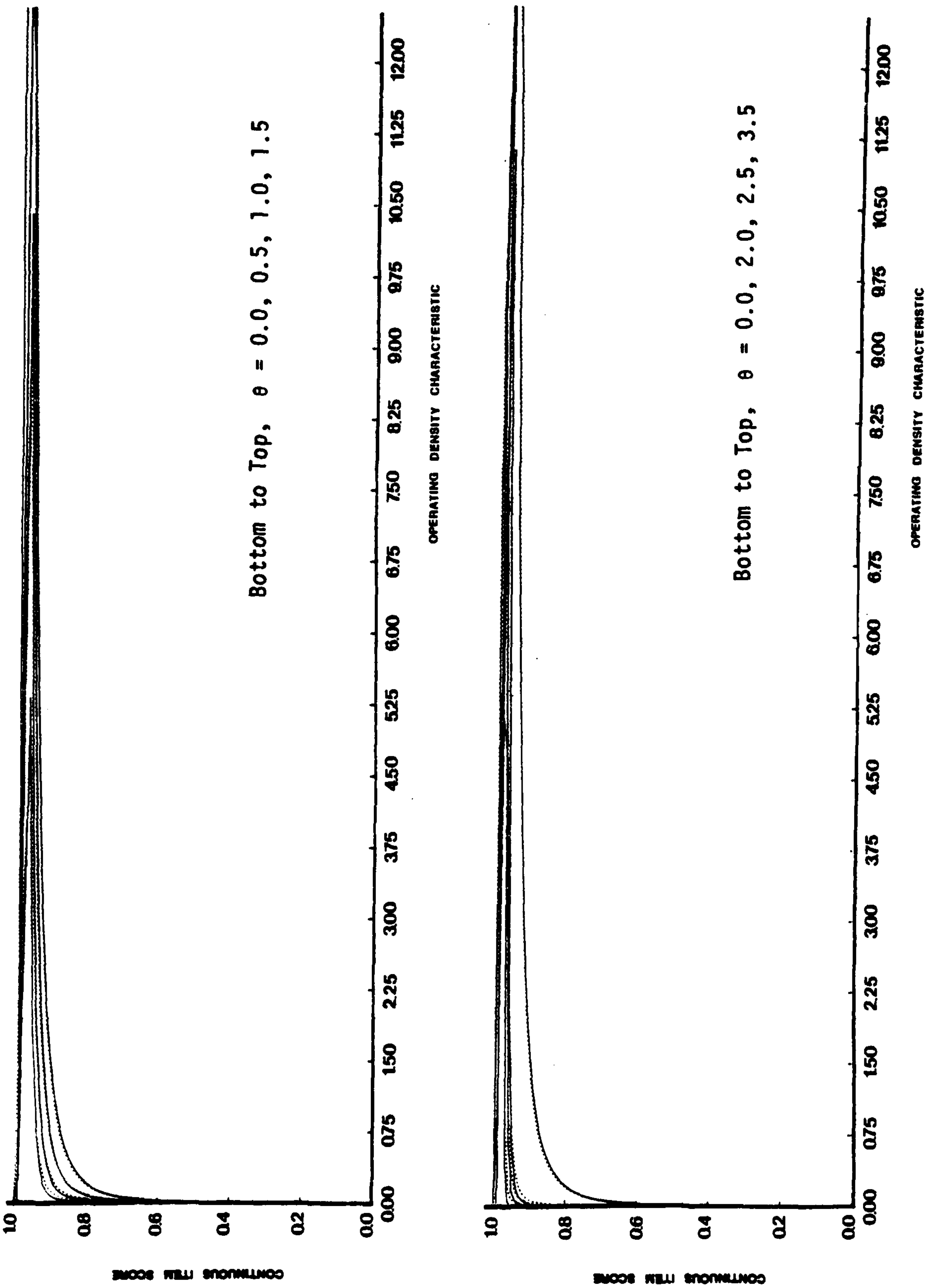


FIGURE A-2-5 (Continued)

Normal Ogive Model (solid line) and Logistic Model (dotted line)

APPENDIX III

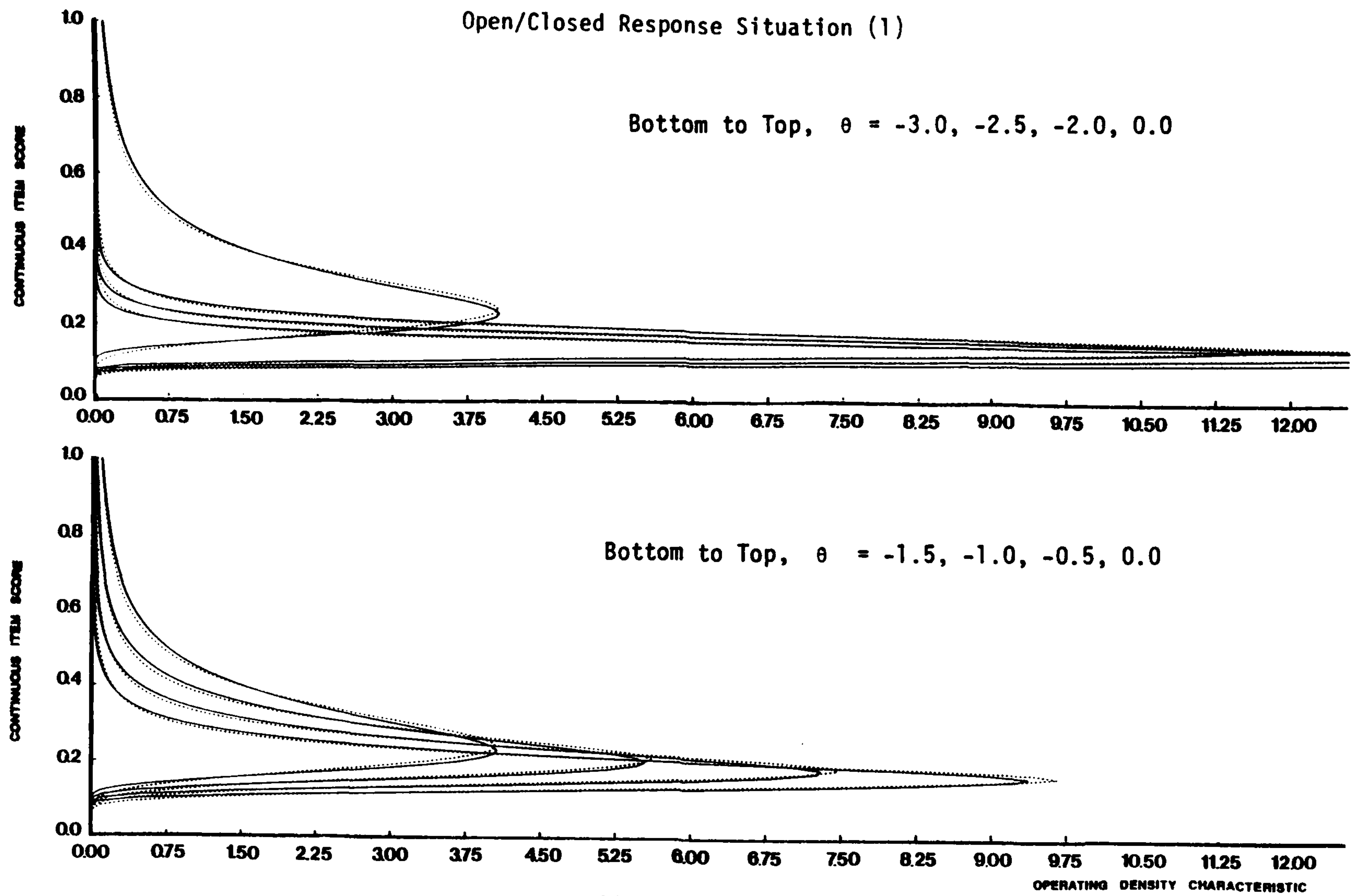


FIGURE A-3-1

Operating Density Characteristic, $H_{z_g}(\theta)$, for Each of the Thirteen Different Values of θ in the Open/Closed Response Situation, in the Normal Ogive Model (solid line) and in the Logistic Model (dotted line), where $a_g = 1.0$ and $D = 1.7$. b_{z_g} is given as a function of Z_g by:

$$b_{z_g} = 2.0 + \tan\{(-\pi/2)[1.0(1-z_g)]\} .$$

APPENDIX III (Continued)

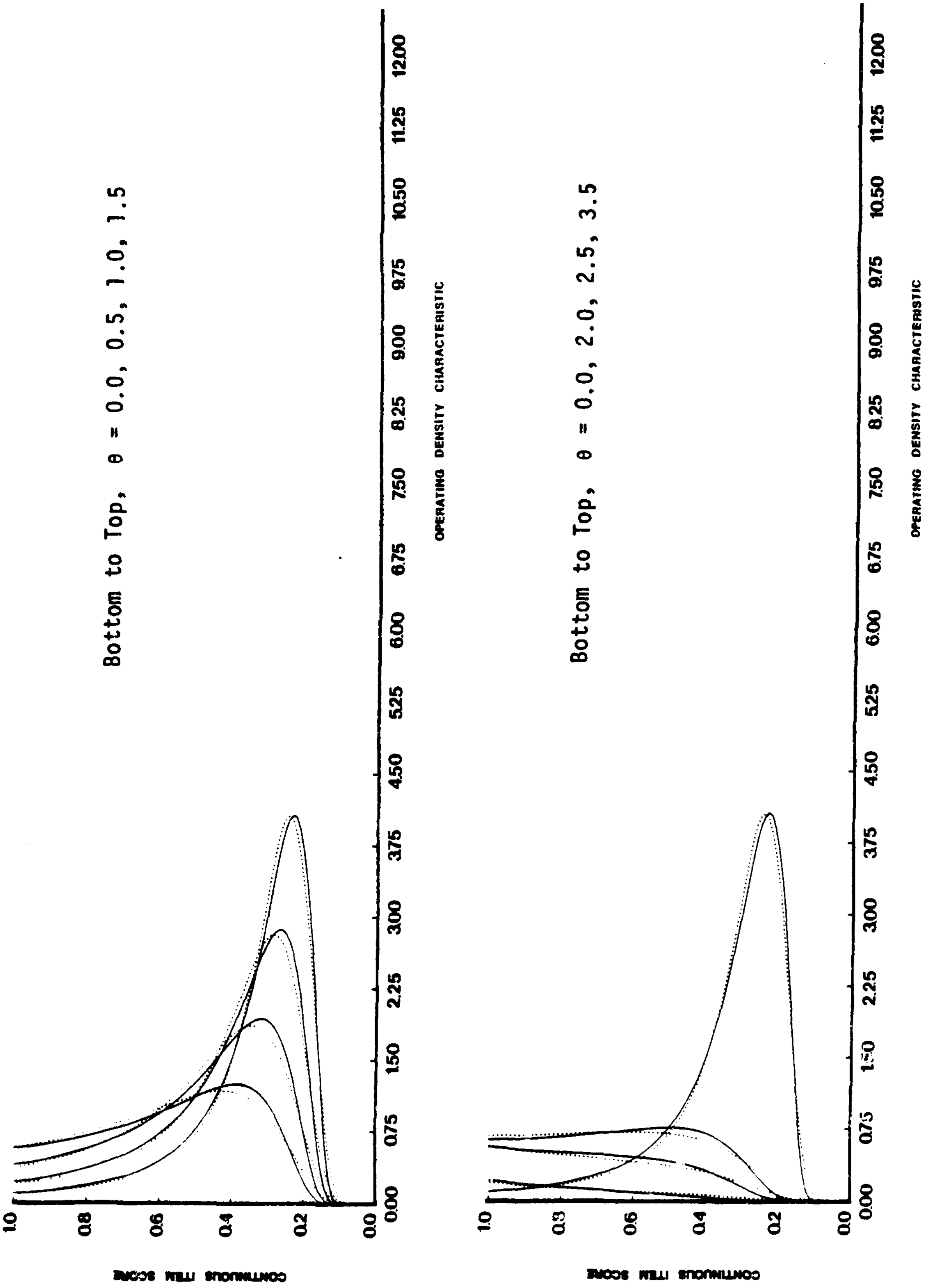


FIGURE A-3-1 (Continued)

Normal Ogive Model (solid line) and Logistic Model (dotted line)

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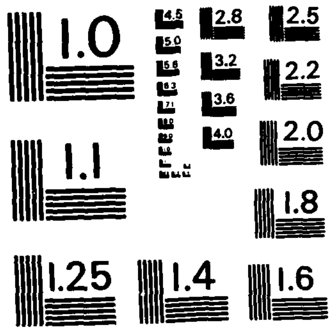

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APPENDIX III (Continued)

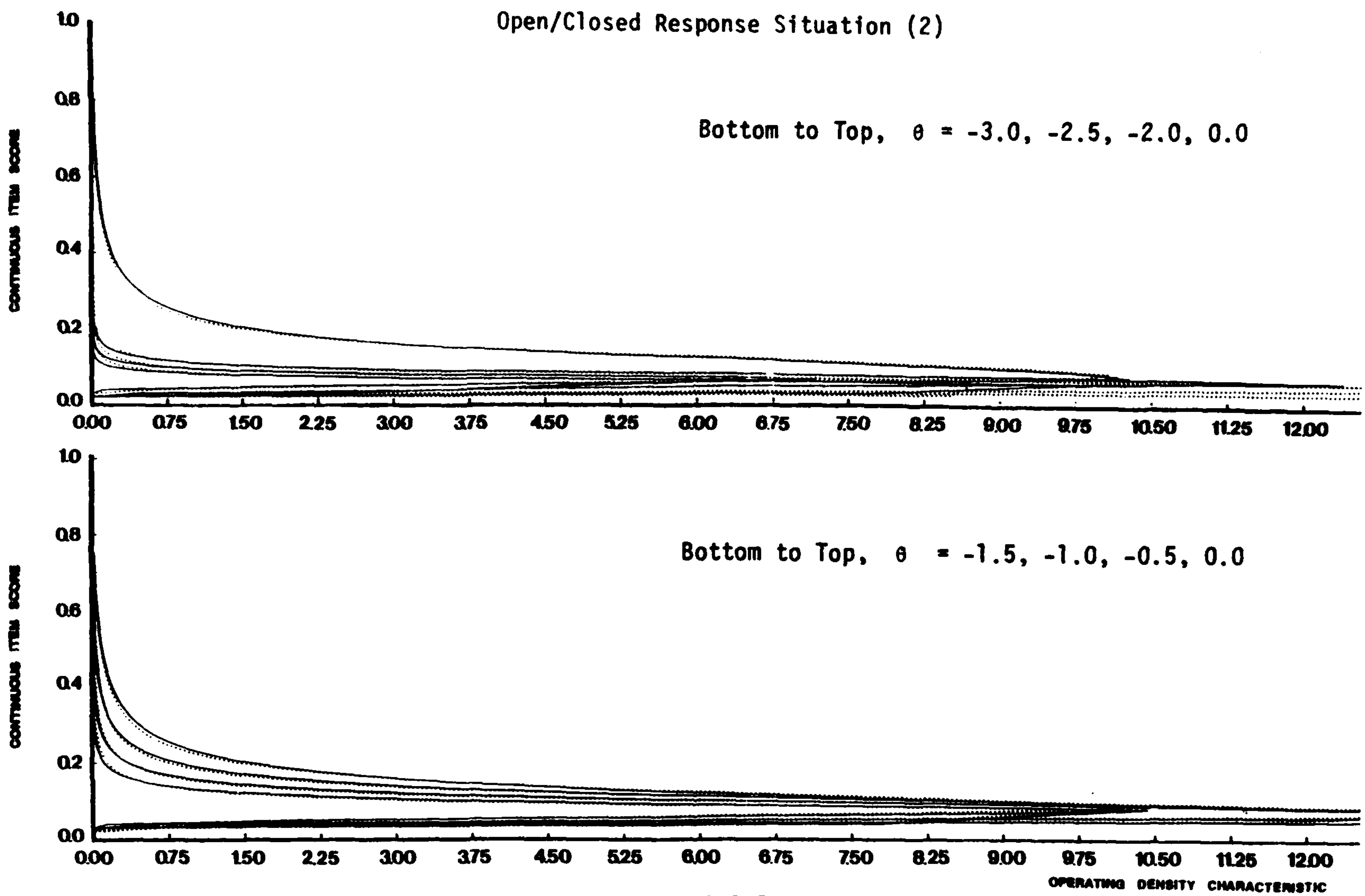


FIGURE A-3-2

Operating Density Characteristic, $H_{z_g}(\theta)$, for Each of the Thirteen Different Values of θ in the Open/Closed Response Situation, in the Normal Ogive Model (solid line) and in the Logistic Model (dotted line), where $a_g = 1.0$ and $D = 1.7$. b_{z_g} is given as a function of Z_g by:

$$b_{z_g} = 2.0 + \tan\{(-\pi/2)[1.0(1-z_g)^3]\}.$$

APPENDIX III (Continued)

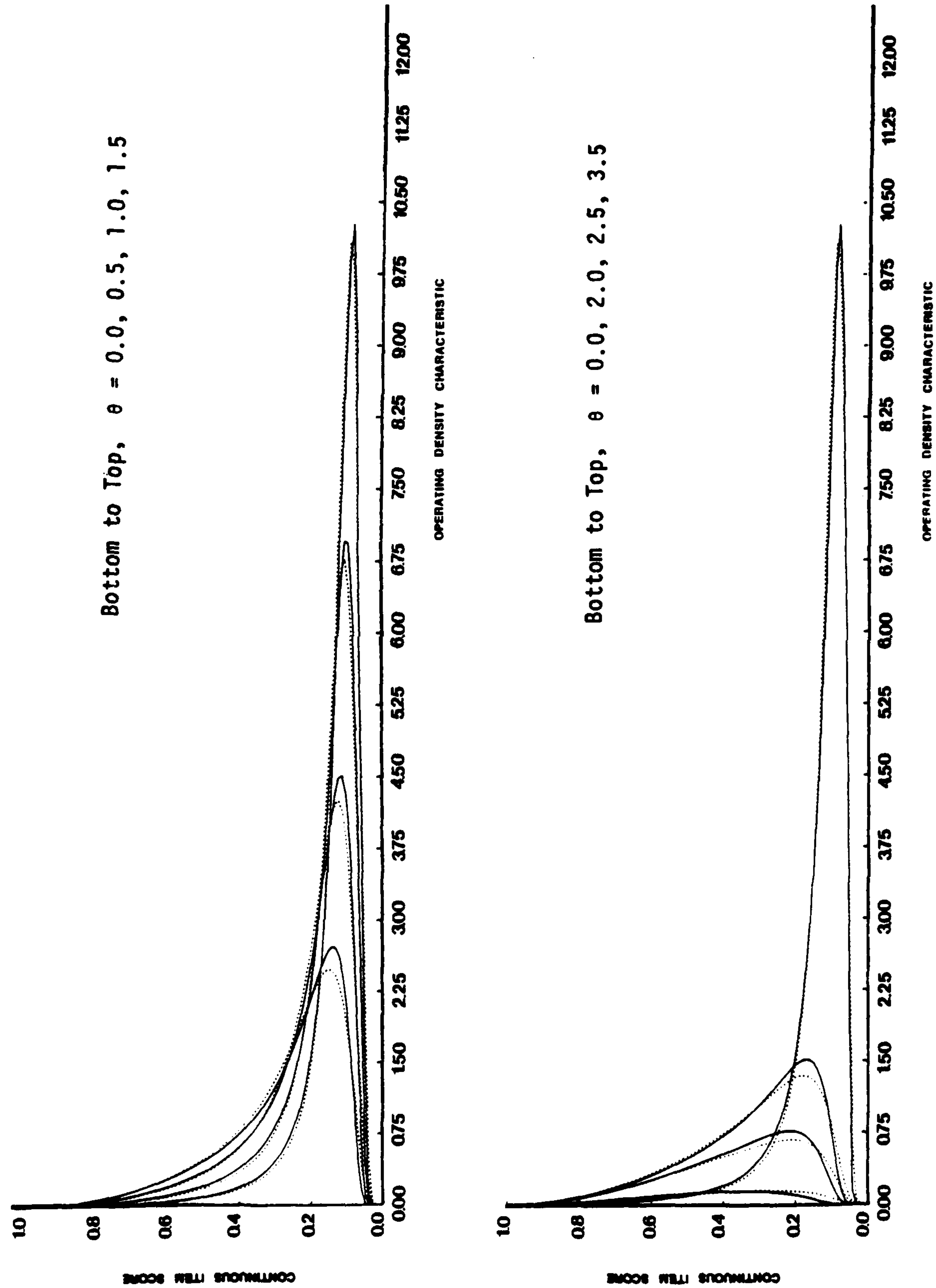


FIGURE A-3-2 (Continued)

Normal Ogive Model (solid line) and Logistic Model (dotted line)

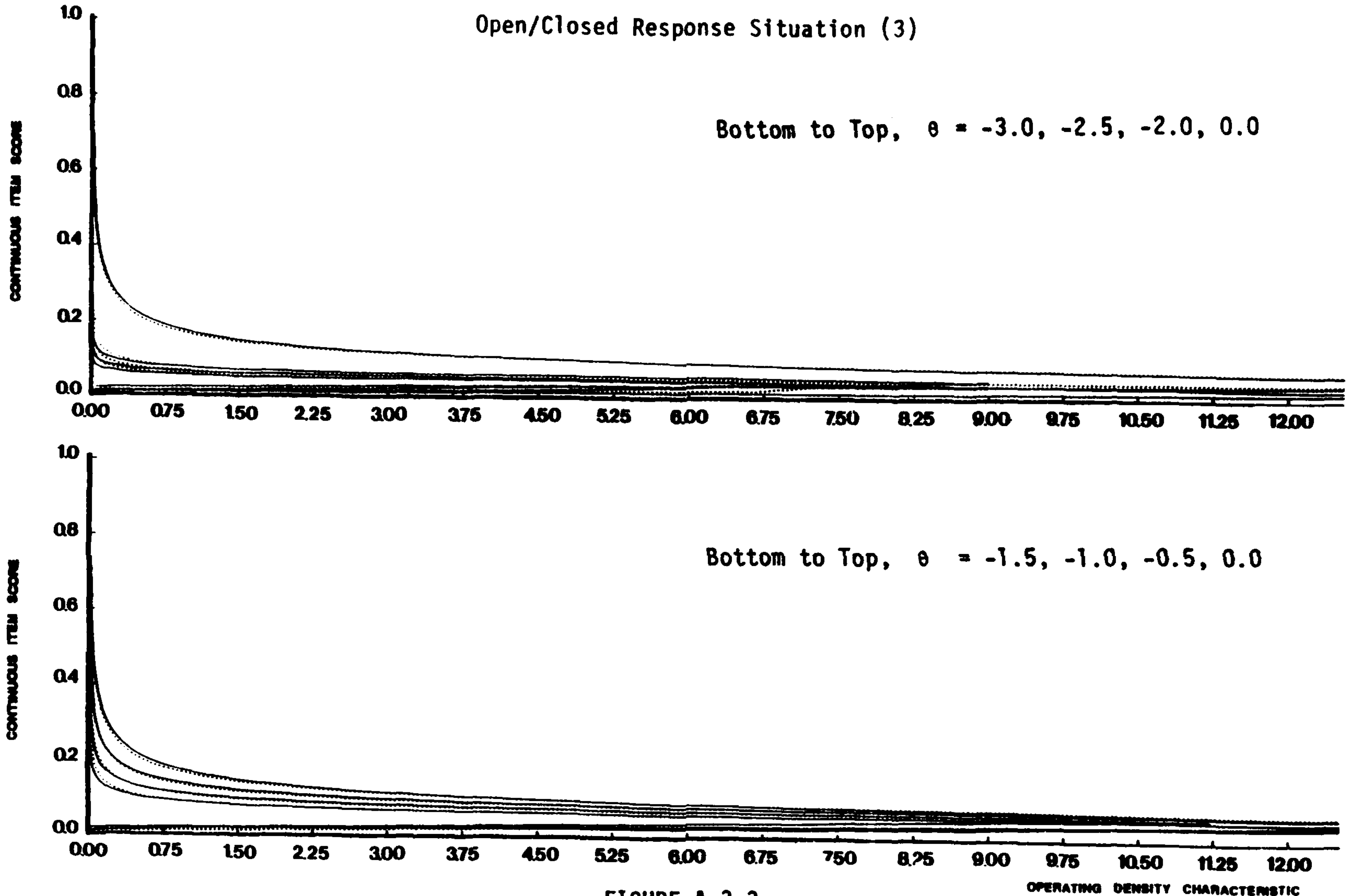
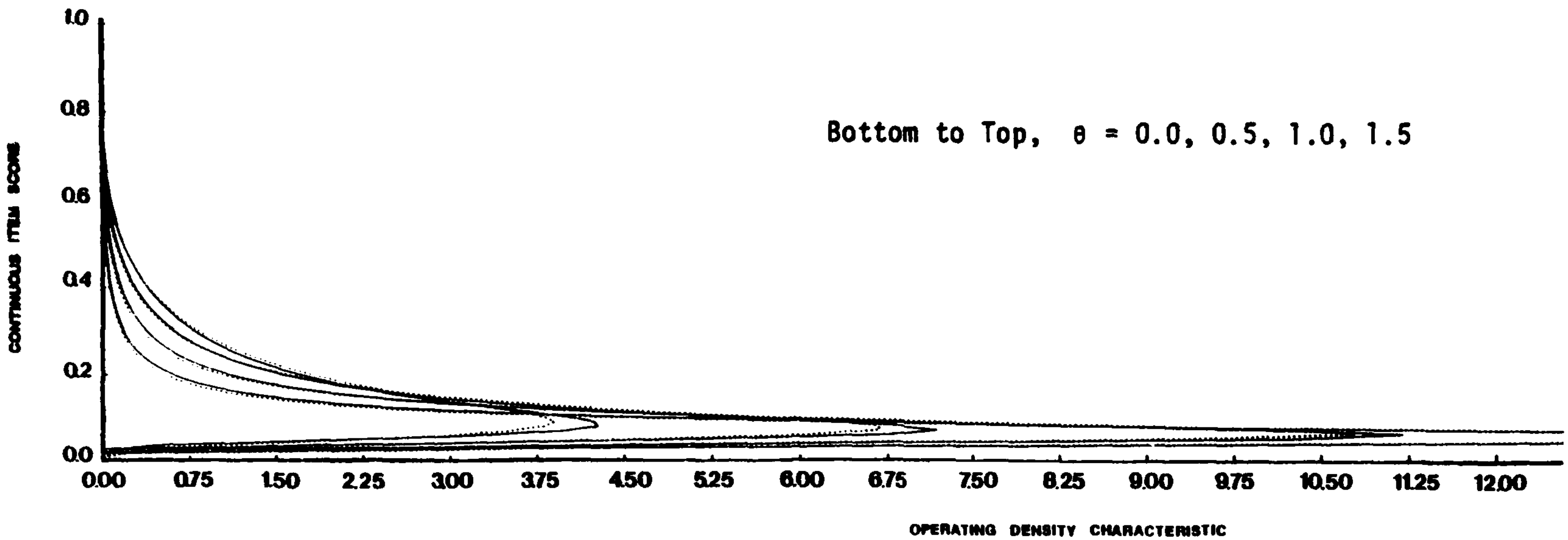


FIGURE A-3-3

Operating Density Characteristic, $H_{z_g}(\theta)$, for Each of the Thirteen Different Values of θ in the Open/Closed Response Situation, in the Normal Ogive Model (solid line) and in the Logistic Model (dotted line), where $a_g = 1.0$ and $D = 1.7$. b_{z_g} is given as a function of z_g by:

$$b_{z_g} = 2.0 + \tan\{(-\pi/2)[1.0(1-z_g)^5]\}$$

APPENDIX III (Continued)



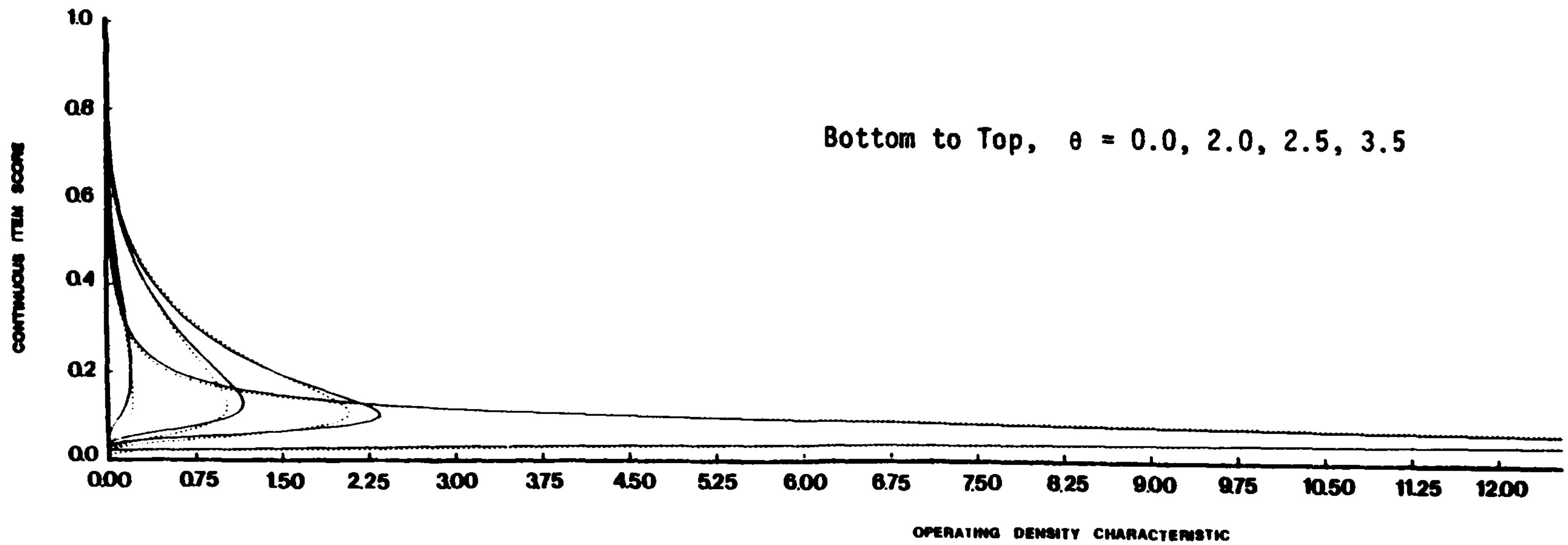


FIGURE A-3-3 (Continued)

Normal Ogive Model (solid line) and Logistic Model (dotted line)

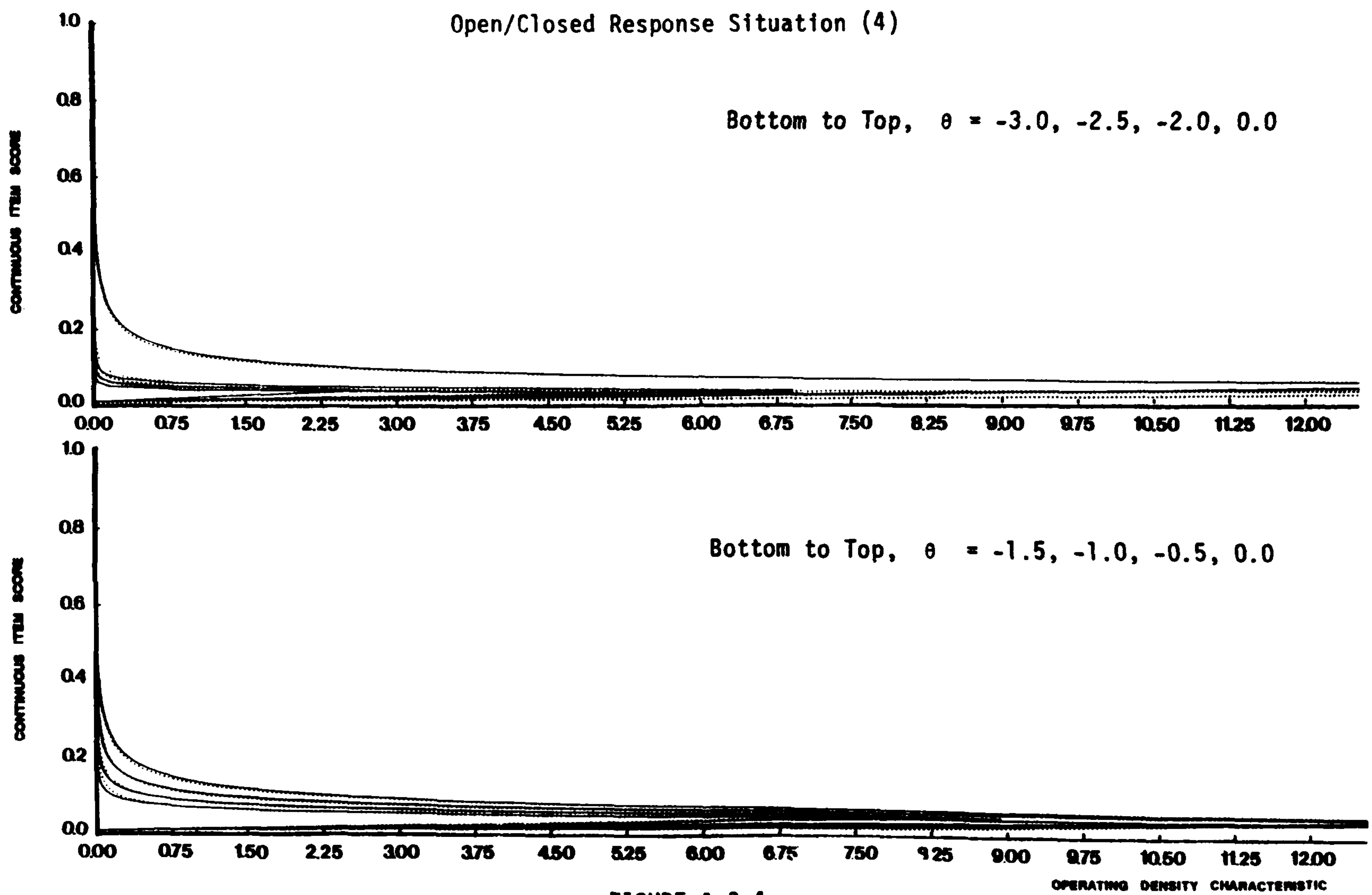
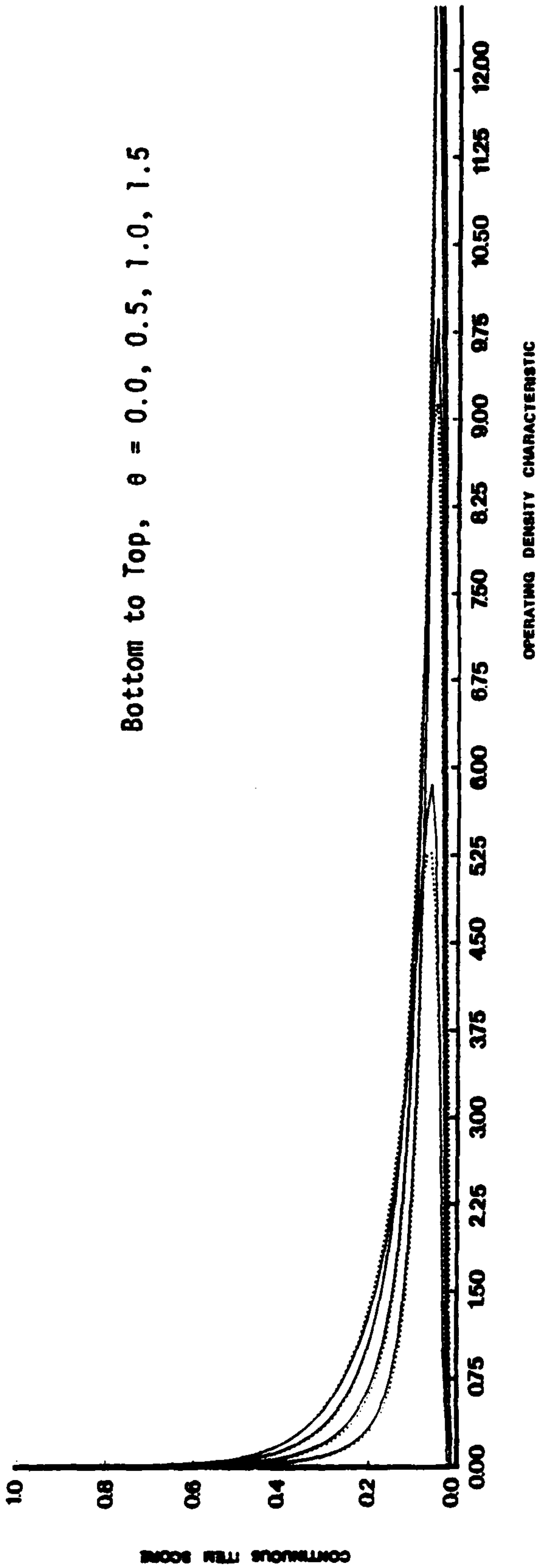


FIGURE A-3-4

Operating Density Characteristic, $H_{z_g}(\theta)$, for Each of the Thirteen Different Values of θ in the Open/Closed Response Situation, in the Normal Ogive Model (solid line) and in the Logistic Model (dotted line), where $a_g = 1.0$ and $D = 1.7$. b_{z_g} is given as a function of Z_g by:

$$b_{z_g} = 2.0 + \tan\{(-\pi/2)[1.0(1-z_g)^7]\}.$$

APPENDIX III (Continued)



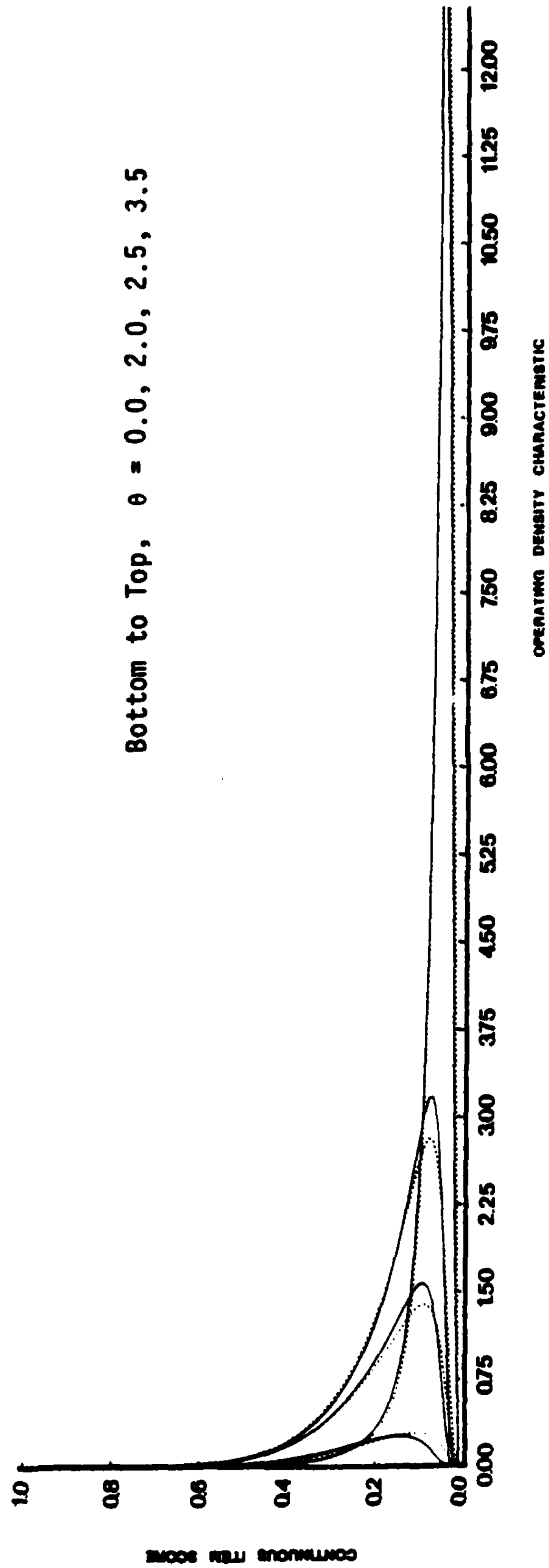


FIGURE A-3-4 (Continued)

Normal Ogive Model (solid line) and Logistic Model (dotted line)

APPENDIX III (Continued)

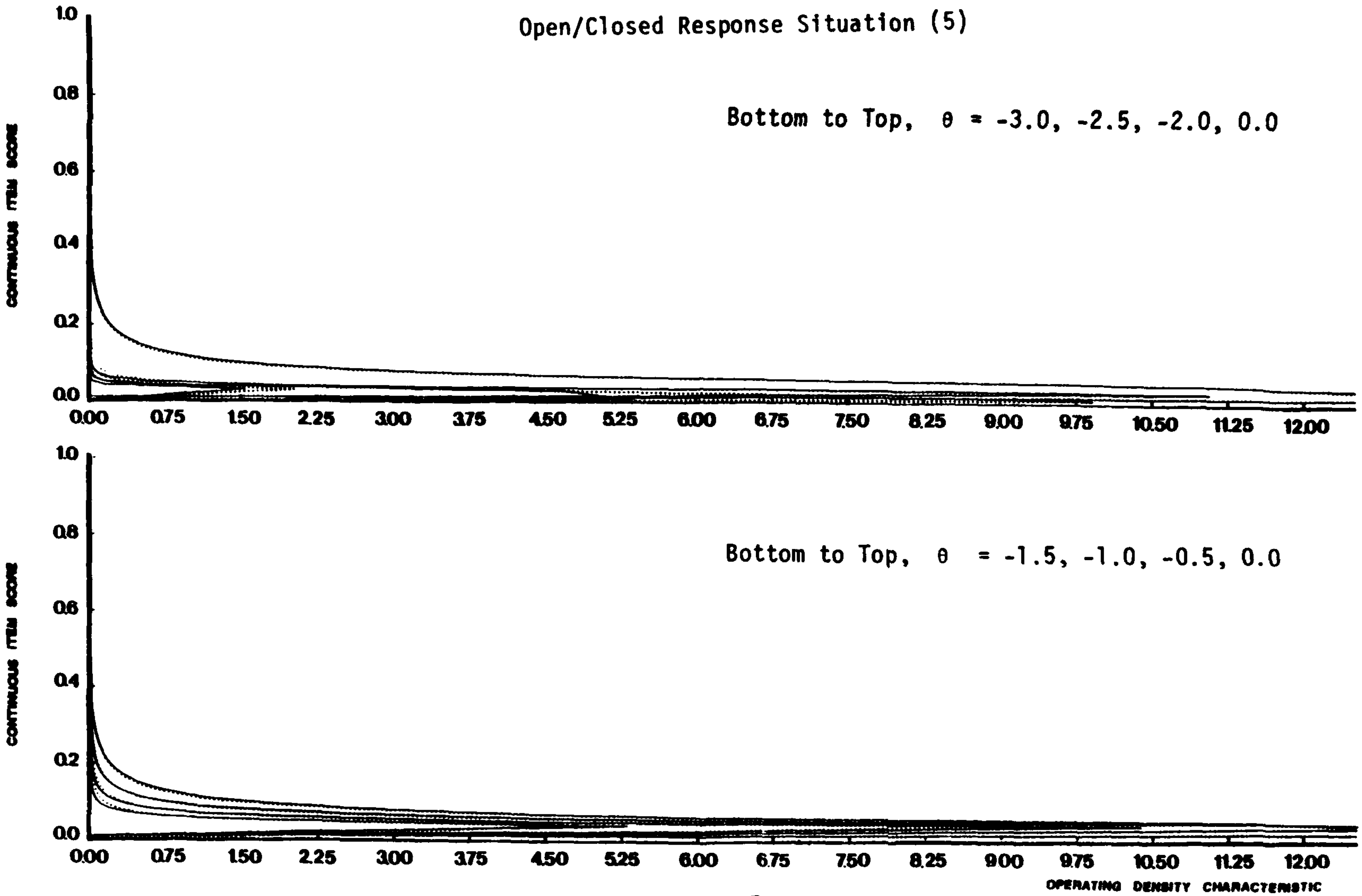
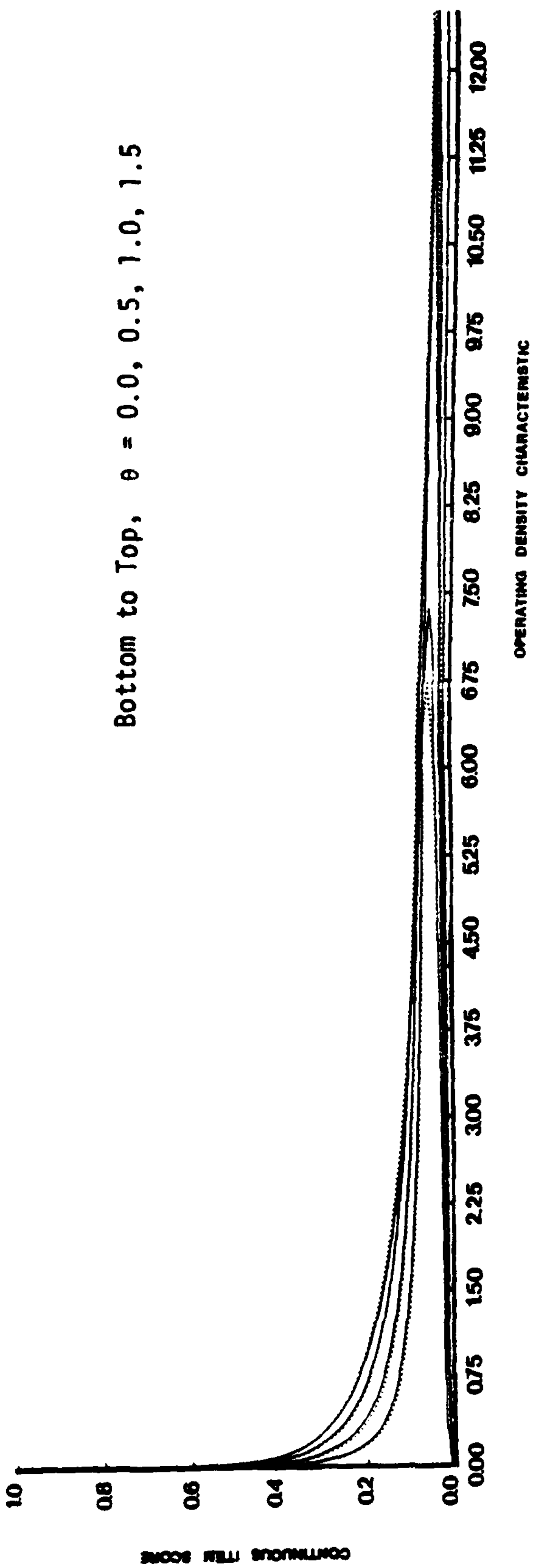


FIGURE A-3-5

Operating Density Characteristic, $H_{z_g}(\theta)$, for Each of the Thirteen Different Values of θ in the Open/Closed Response Situation, in the Normal Ogive Model (solid line) and in the Logistic Model (dotted line), where $a_g = 1.0$ and $D = 1.7$. b_{z_g} is given as a function of z_g by:

$$b_{z_g} = 2.0 + \tan\{(-\pi/2)[1.0(1-z_g)^9]\}.$$

APPENDIX III (Continued)



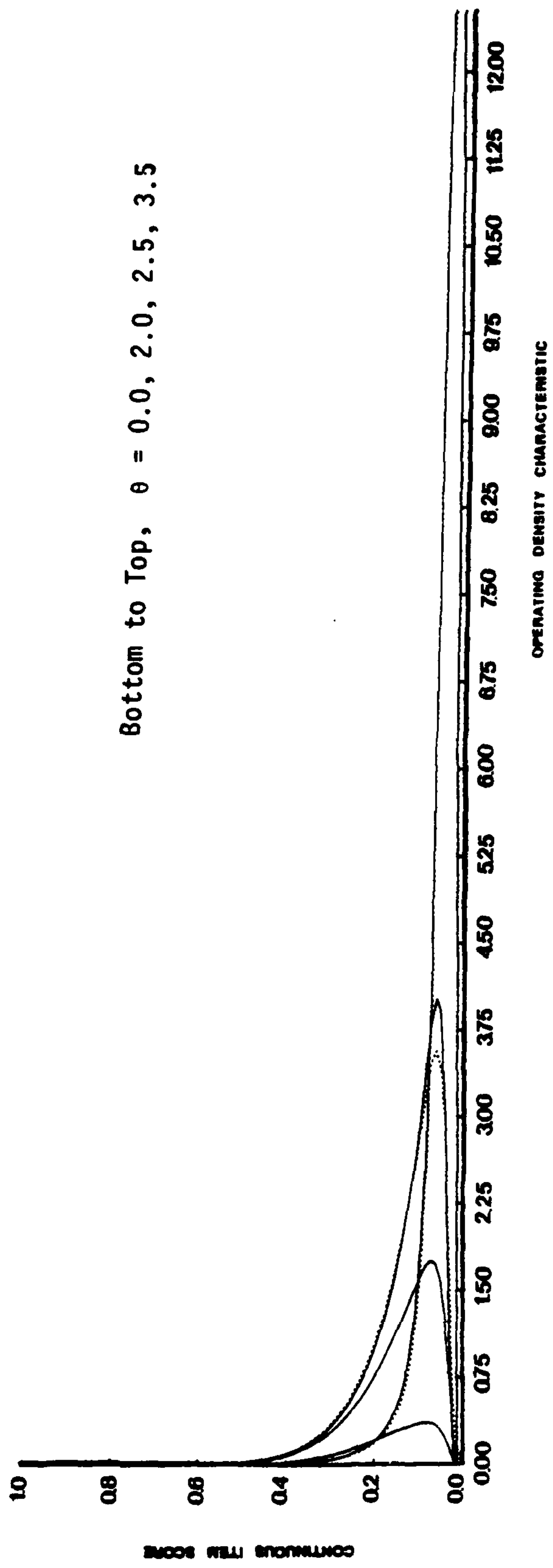


FIGURE A-3-5 (Continued)

Normal Ogive Model (solid line) and Logistic Model (dotted line)

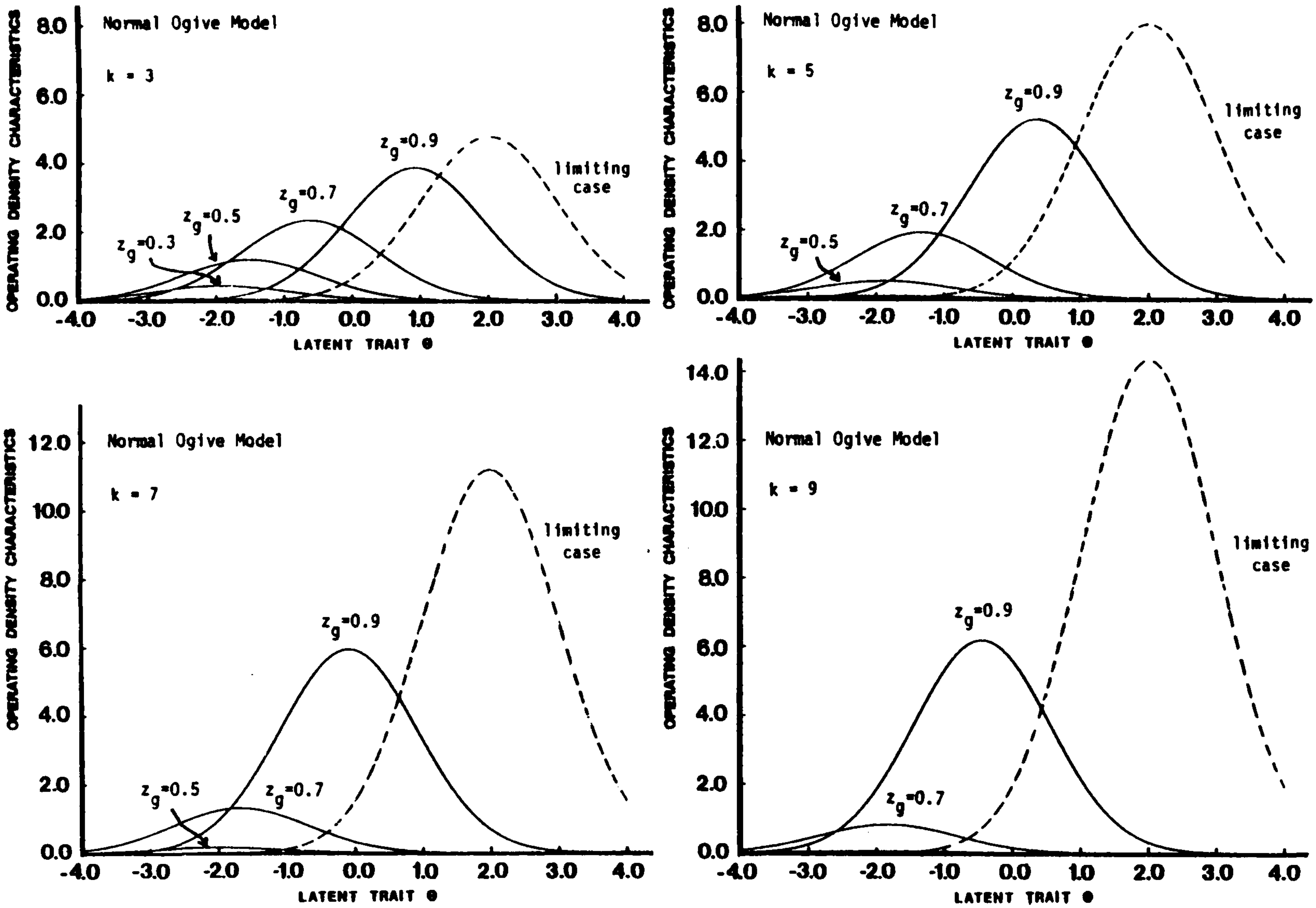


FIGURE A-4

Operating Density Characteristic, $H_{z_g}(\theta)$, As a Function of θ for Each of the Five Values of the Item Score, 0.1, 0.3, 0.5, 0.7 and 0.9, Following the Normal Ogive and the Logistic Models, with $a_g = 1.0$, $b_0 = -2.0$ and $D = 1.7$, When the Functional Relationship between the Item Score z_g and the Difficulty Parameter b_{z_g} Is Given by $b_{z_g} = b_0 + (b_1 - b_0) z_g^k$ $(k = 3, 5, 7, 9)$. The Additional Two Curves Are Those in the Limiting Situations Where z_g Tends to Zero and Unity, Respectively. Closed Response Situation.

APPENDIX IV (Continued)

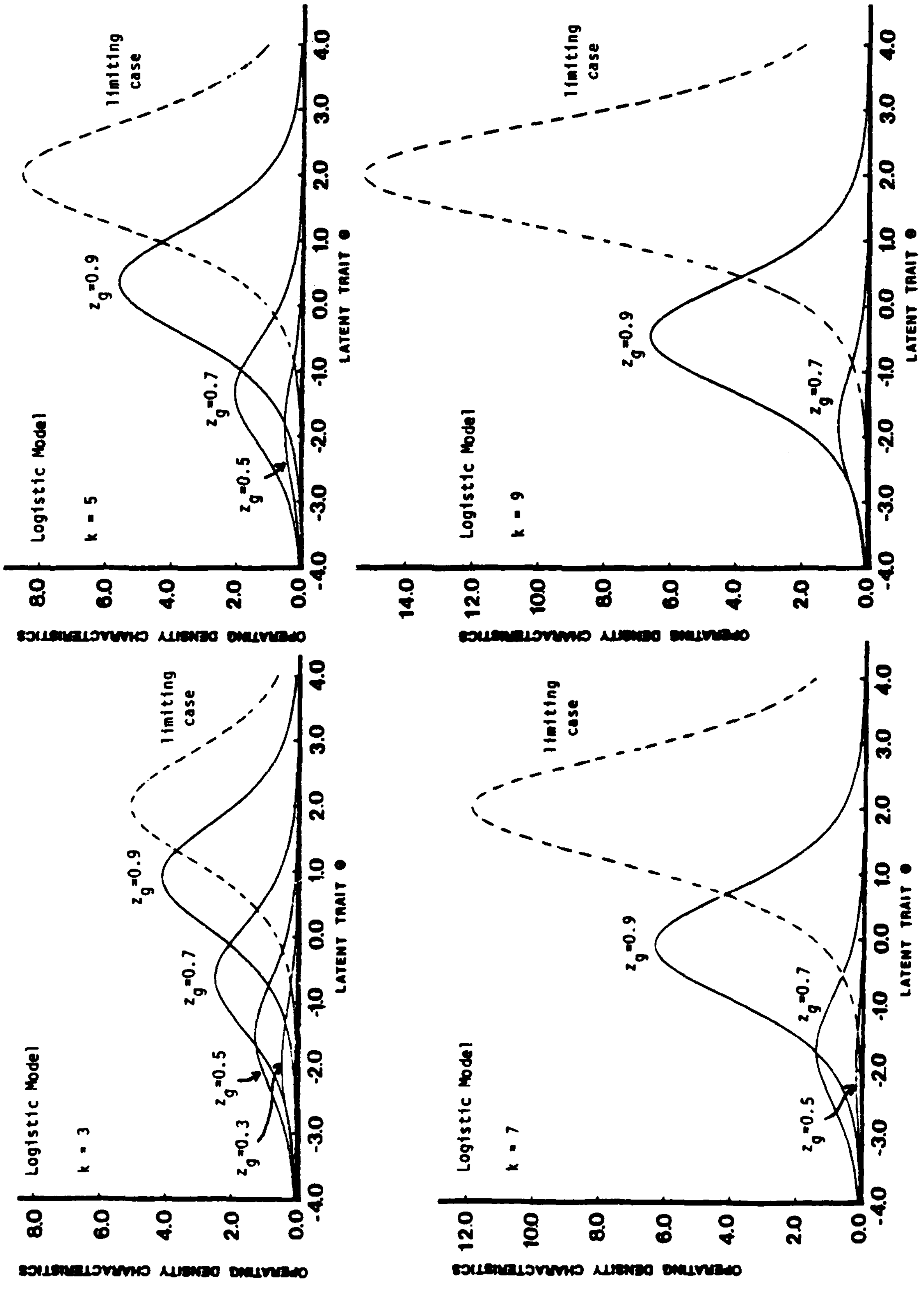


FIGURE A-4 (Continued)

APPENDIX V

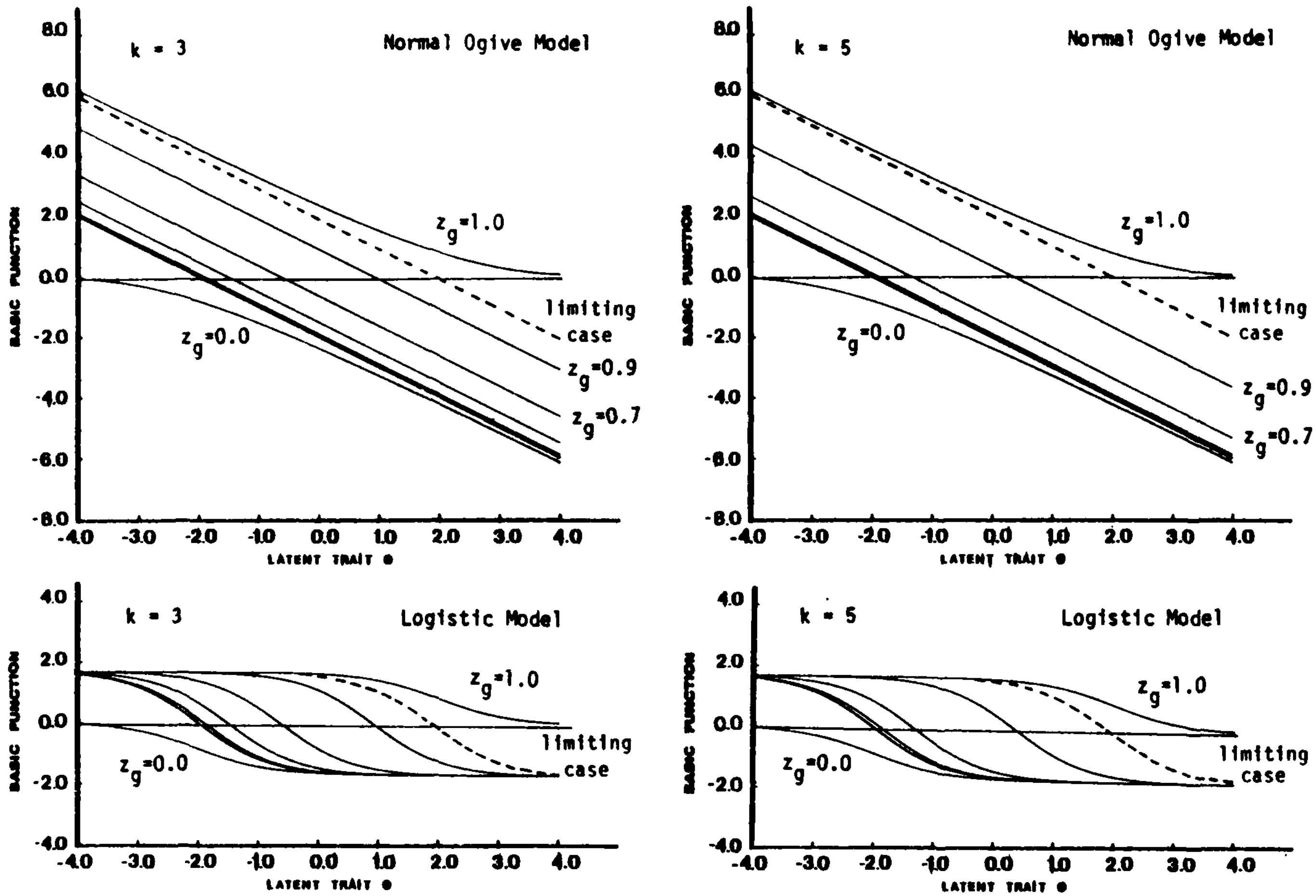


FIGURE A-5

Basic Function, $A_{z_g}(\theta)$, for Each of the Seven Values of the Item Score, 0.0, 0.1, 0.3, 0.5, 0.7, 0.9 and 1.0, Following the Normal Ogive and the Logistic Models, with $a_g = 1.0$, $b_0 = -2.0$, $b_1 = 2.0$ and $D = 1.7$, When the Functional Relationship between the Item Score and the Difficulty Parameter b_{z_g} Is Given by $b_{z_g} = b_0 + (b_1 - b_0)z_g$ $(k = 3, 5, 7, 9)$. Closed Response Situation.

APPENDIX V (Continued)

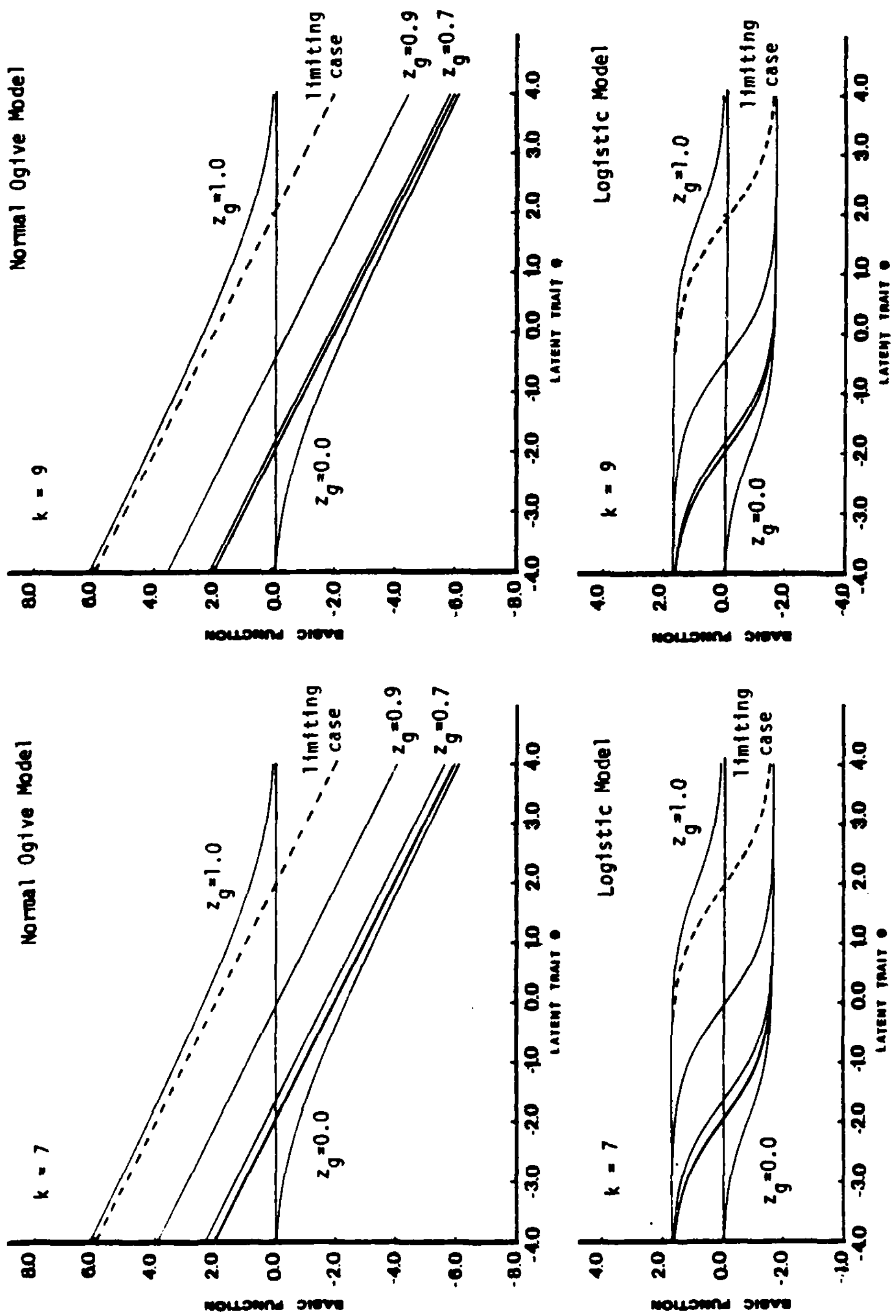


FIGURE A-5 (Continued)

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APPENDIX XII

Logistic Model

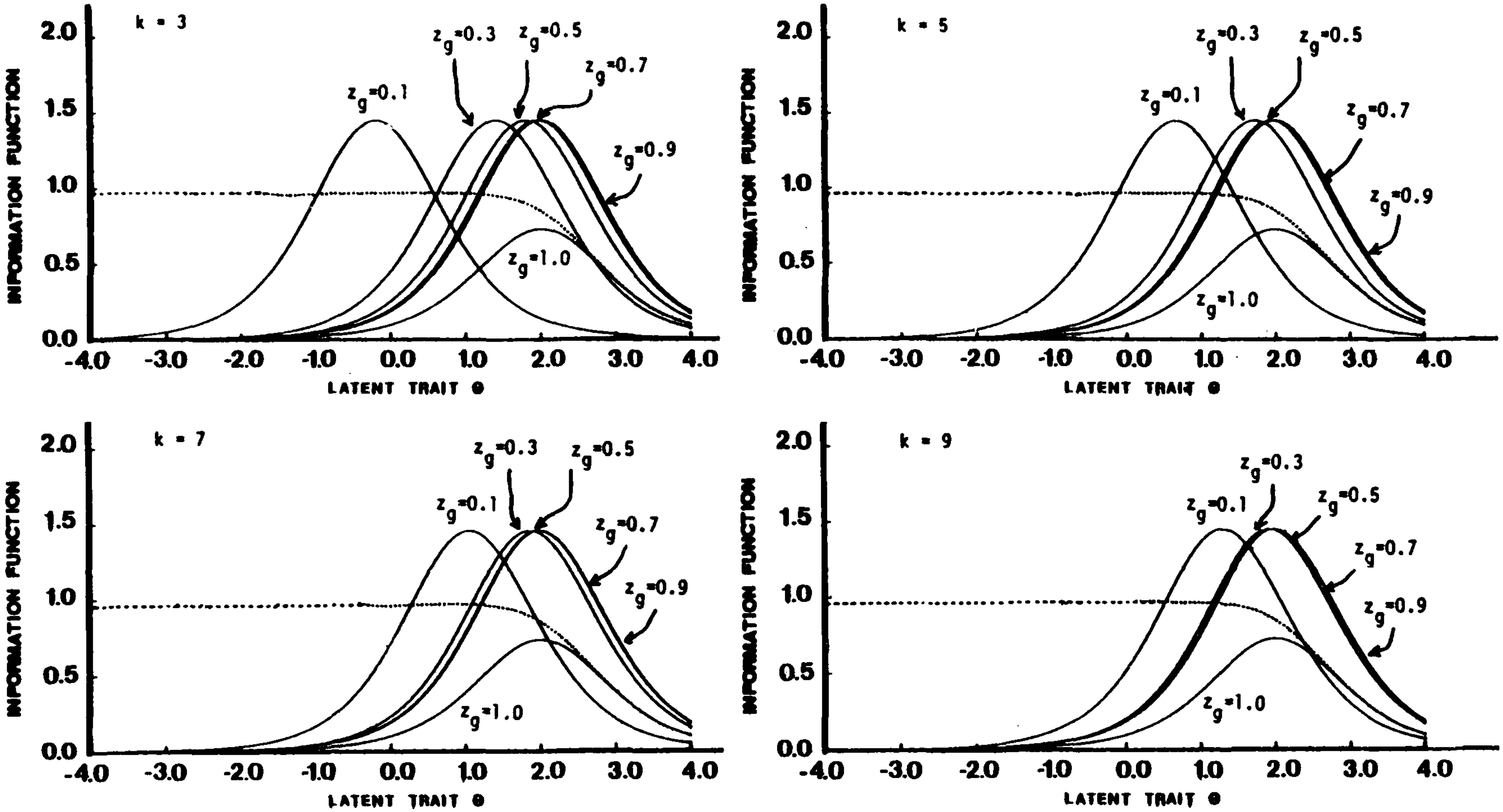

FIGURE A-12

Item Response Information Functions, $I_{z_g}(\theta)$, (Solid Line) and Item Information Function, $I_g(\theta)$, (Dotted Line) in the Logistic Model for z_g = 0.1, 0.3, 0.5, 0.7, 0.9, 1.0 , with $a_g = 1.0$, $b_1 = 2.0$ and $D = 1.7$, When the Functional Relationship between z_g and b_{z_g} Is Given by

$$b_{z_g} = b_1 + \tan[(-\pi/2)(1-z_g)^k]$$ for $k = 3, 5, 7, 9$. Open/Closed Response Situation.

APPENDIX XI (Continued)

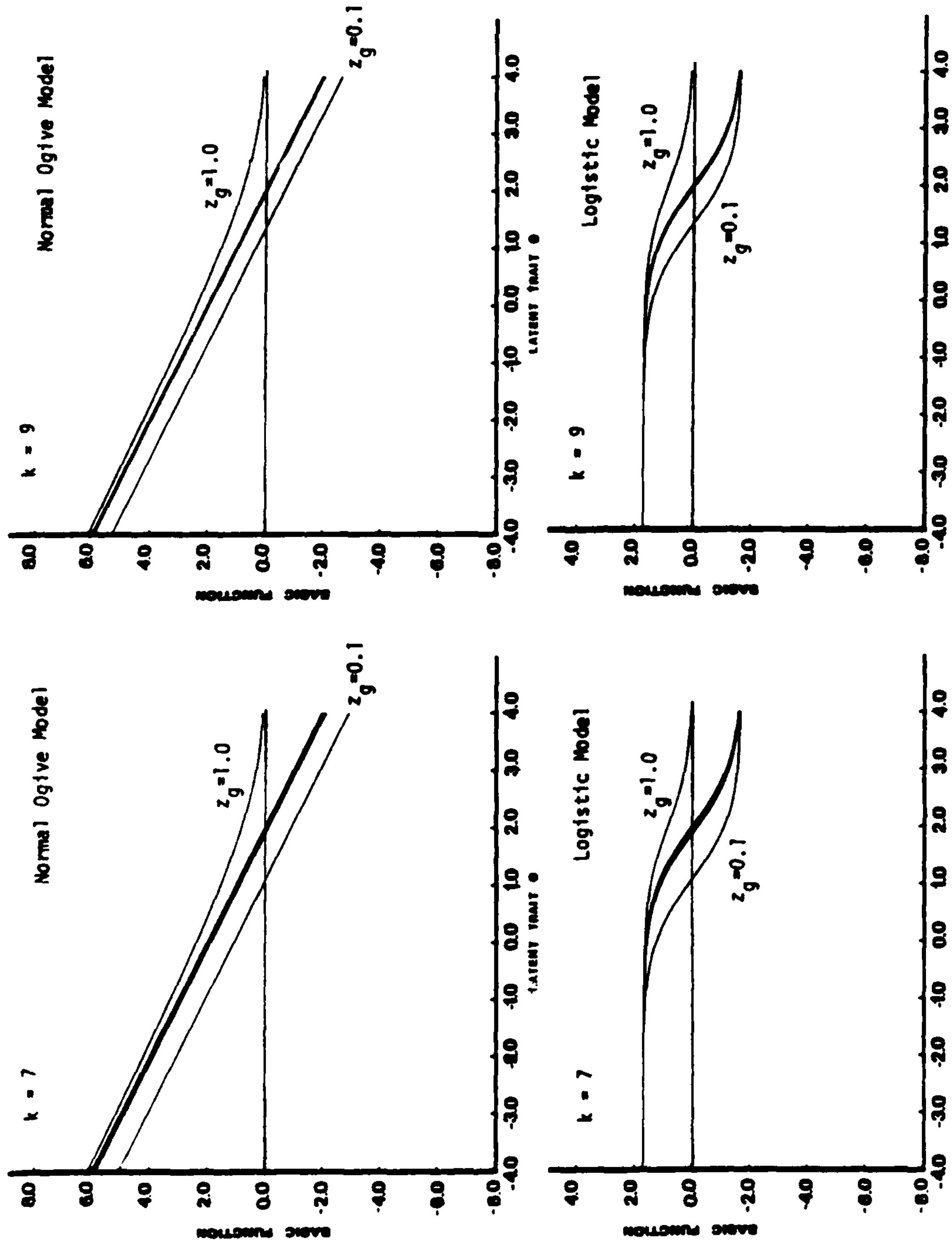

FIGURE A-11 (Continued)

APPENDIX XI

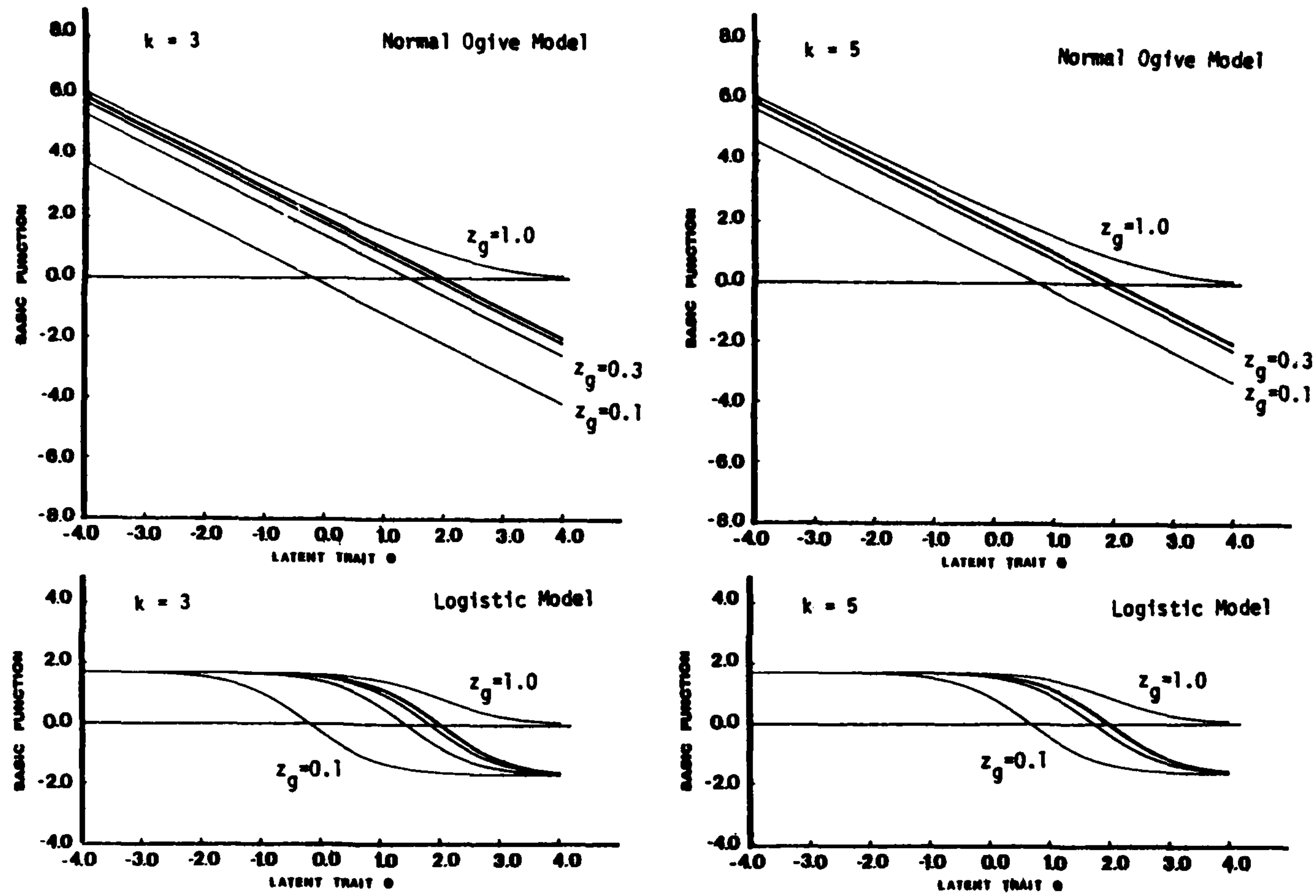

FIGURE A-11

Basic Function, $A_{z_g}(\theta)$, for Each of the Six Values of the Item Score, 0.1, 0.3, 0.5, 0.7, 0.9 and 1.0, Following the Normal Ogive and the Logistic Models, with $a_g = 1.0$, $b_1 = 2.0$ and $D = 1.7$, When the Functional Relationship between the Item Score z_g and the Difficulty Parameter b_{z_g} Is Given by $b_{z_g} = b_1 + \tan[(-\pi/2)(1-z_g)^k]$ $(k = 3, 5, 7, 9)$. Open/Closed Response Situation.

APPENDIX X (Continued)

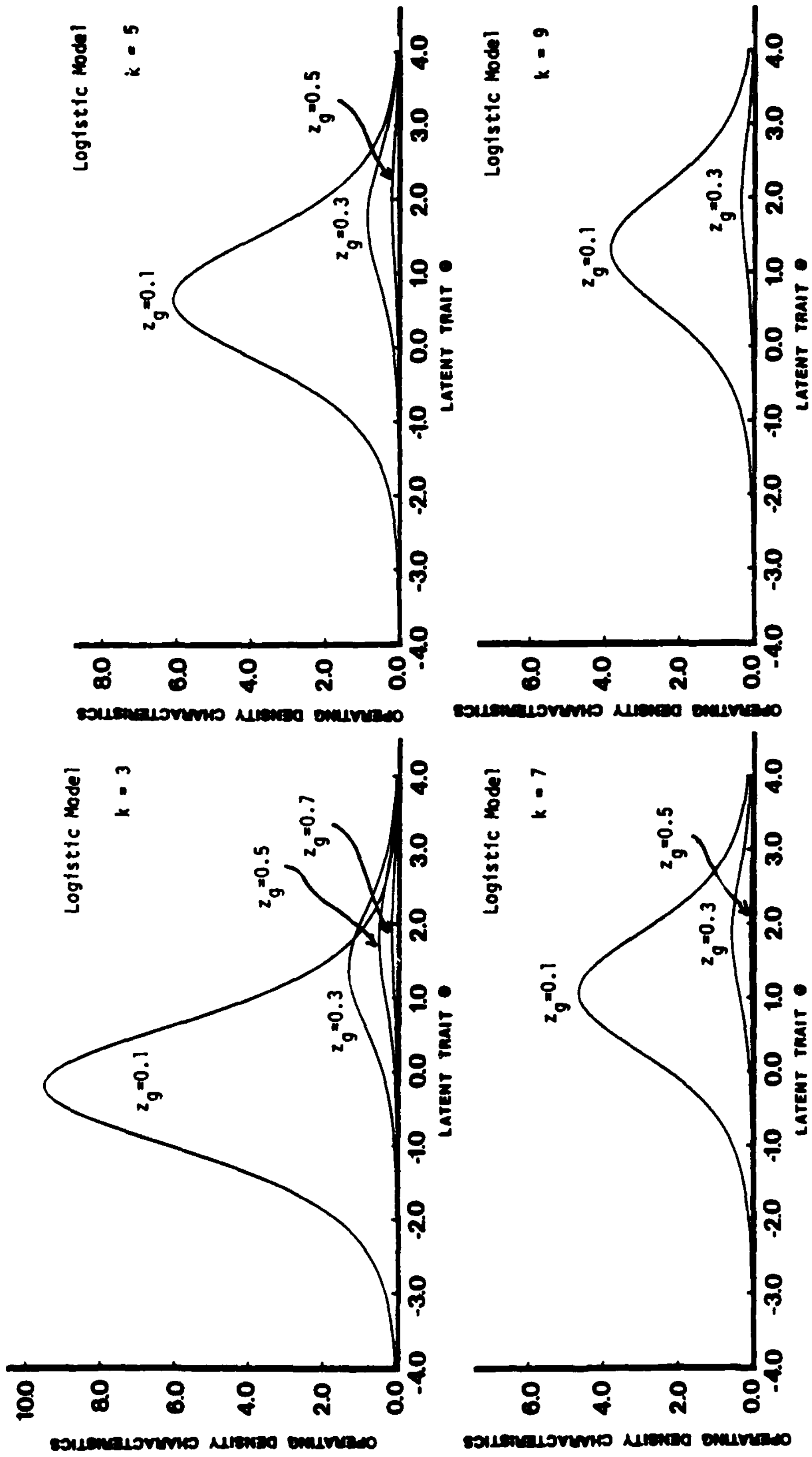

FIGURE A-10 (Continued)

APPENDIX X

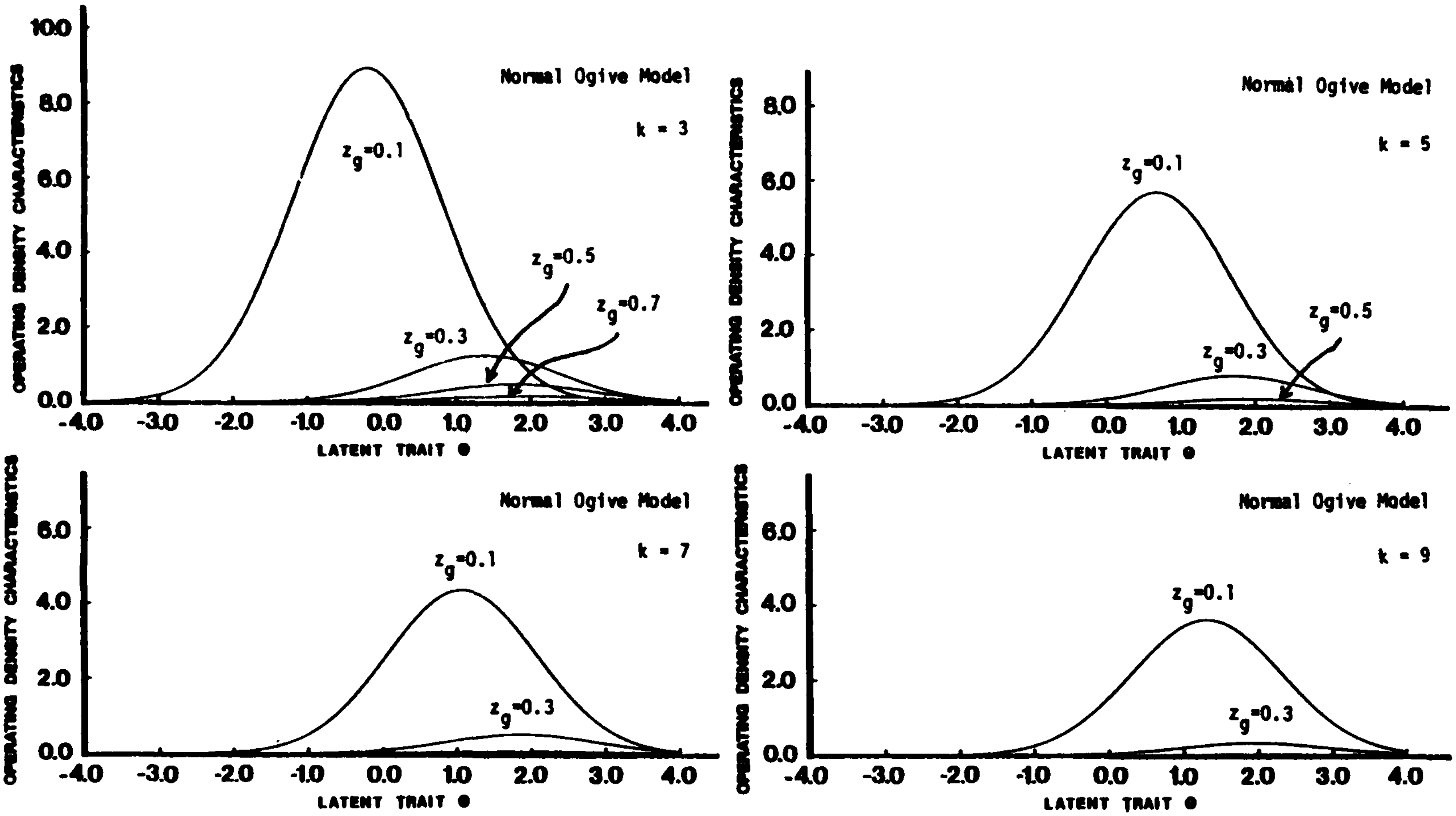

FIGURE A-10

Operating Density Characteristic, $H_{z_g}(\theta)$, As a Function of θ for Each of the Five Values of the Item Score, 0.1 , 0.3 , 0.5 , 0.7 and 0.9 , Following the Normal Ogive and the Logistic Models with $a_g = 1.0$, $b_1 = 2.0$ and $D = 1.7$, When the Functional Relationship between the Item Score z_g and the Difficulty Parameter b_{z_g} Is Given by $b_{z_g} = b_1 + \tan[(-\pi/2)(1-z_g)^k]$ $(k = 3, 5, 7, 9)$. The Additional Curve Is the One in the Limiting Situation Where z_g Tends to Unity. Open/Closed Response Situation.

APPENDIX IX

Logistic Model

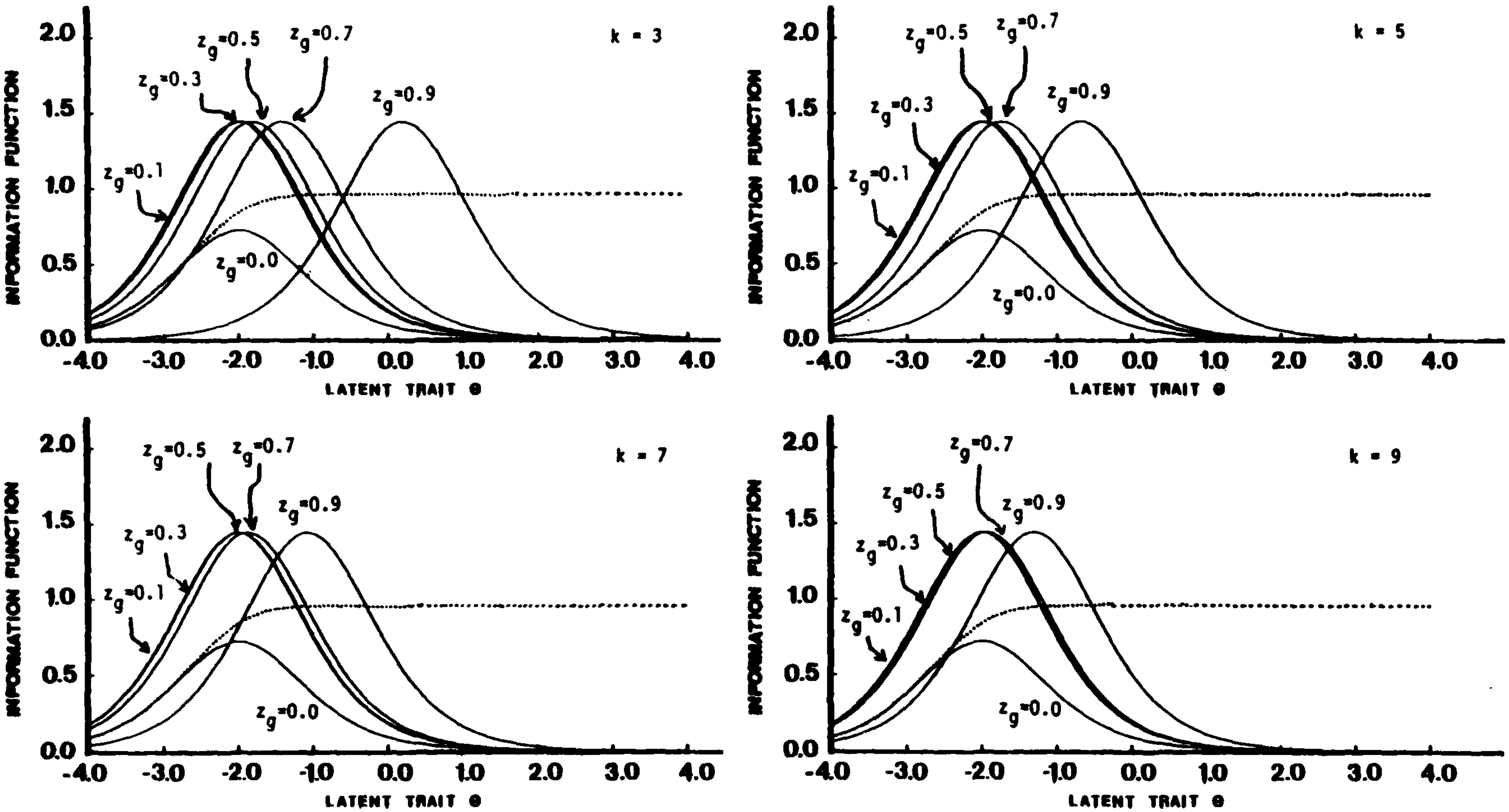

FIGURE A-9

Item Response Information Functions, $I_{z_g}(\theta)$, (Solid Line) and Item Information Function, $I_g(\theta)$, (Dotted Line) in the Logistic Model for z_g = 0.0, 0.1, 0.3, 0.5, 0.7, 0.9 , with $a_g = 1.0$, $b_0 = -2.0$ and $D = 1.7$, When the Functional Relationship between z_g and b_{z_g} Is Given by $b_{z_g} = b_0 + \tan[(\pi/2)z_g^k]$ for $k = 3, 5, 7, 9$. Closed/Open Response Situation.

APPENDIX VIII (Continued)

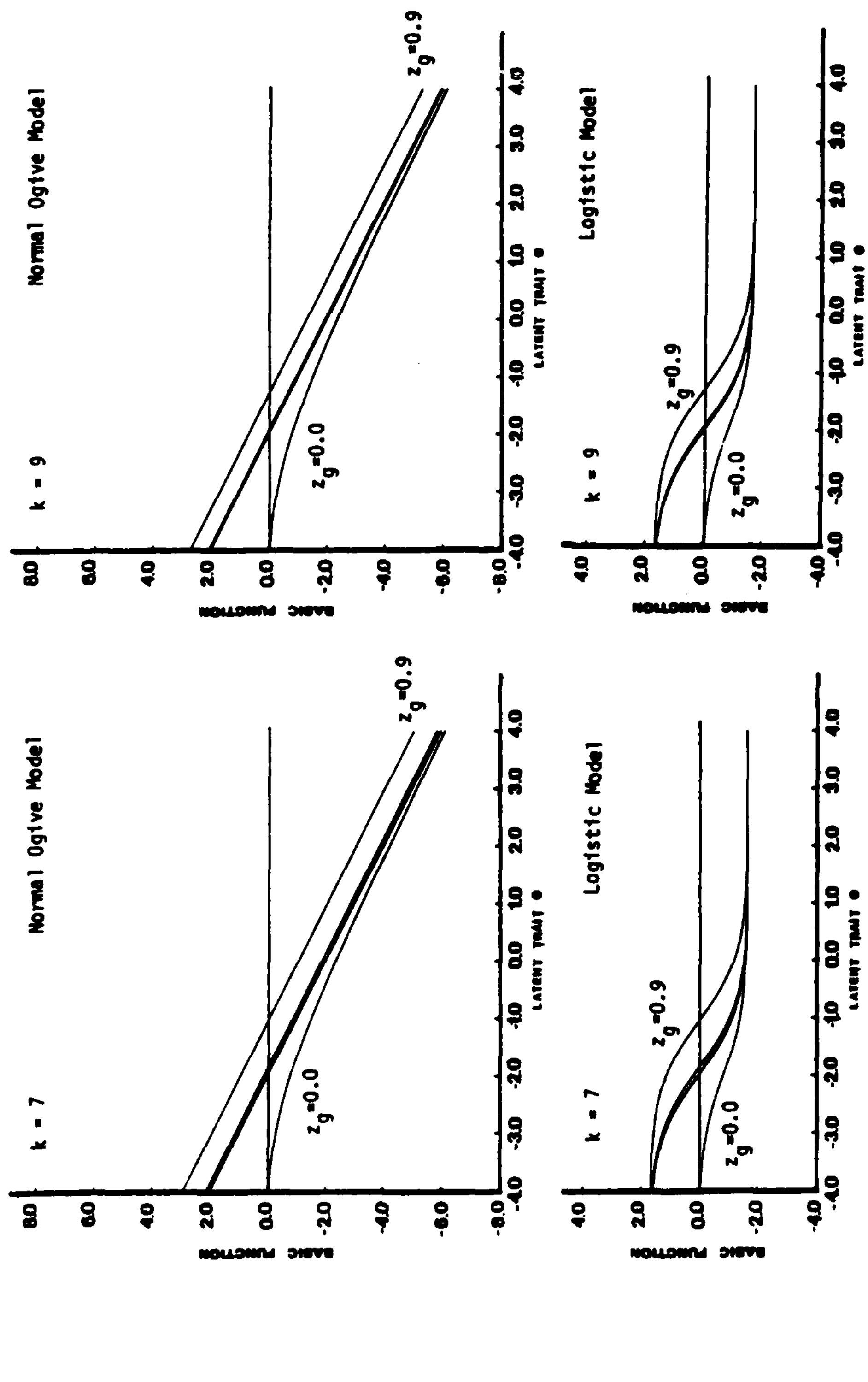

FIGURE A-8 (Continued)

APPENDIX VIII

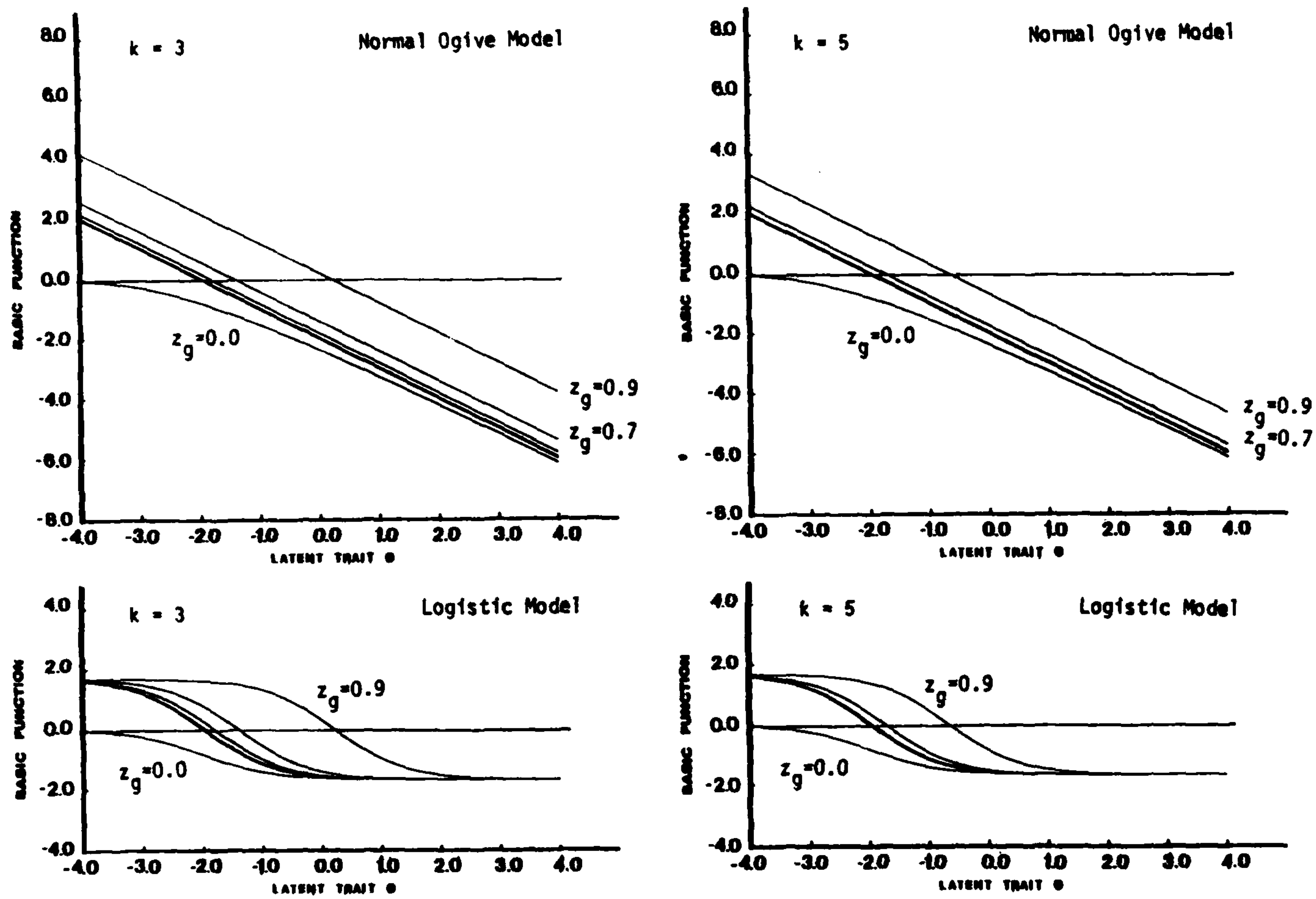

FIGURE A-8

Basic Function, $A_{z_g}(\theta)$, for Each of the Six Values of the Item Score, 0.0, 0.1, 0.3, 0.5, 0.7 and 0.9 , Following the Normal Ogive and the Logistic Models, with $a_g = 1.0$, $b_0 = -2.0$ and $D = 1.7$, When the Functional Relationship between the Item Score z_g and the Difficulty Parameter b_{z_g} Is Given by $b_{z_g} = b_0 + \tan[(\pi/2)z_g^k]$ (k = 3, 5, 7, 9) . Closed/Open Response Situation.

APPENDIX VII (Continued)

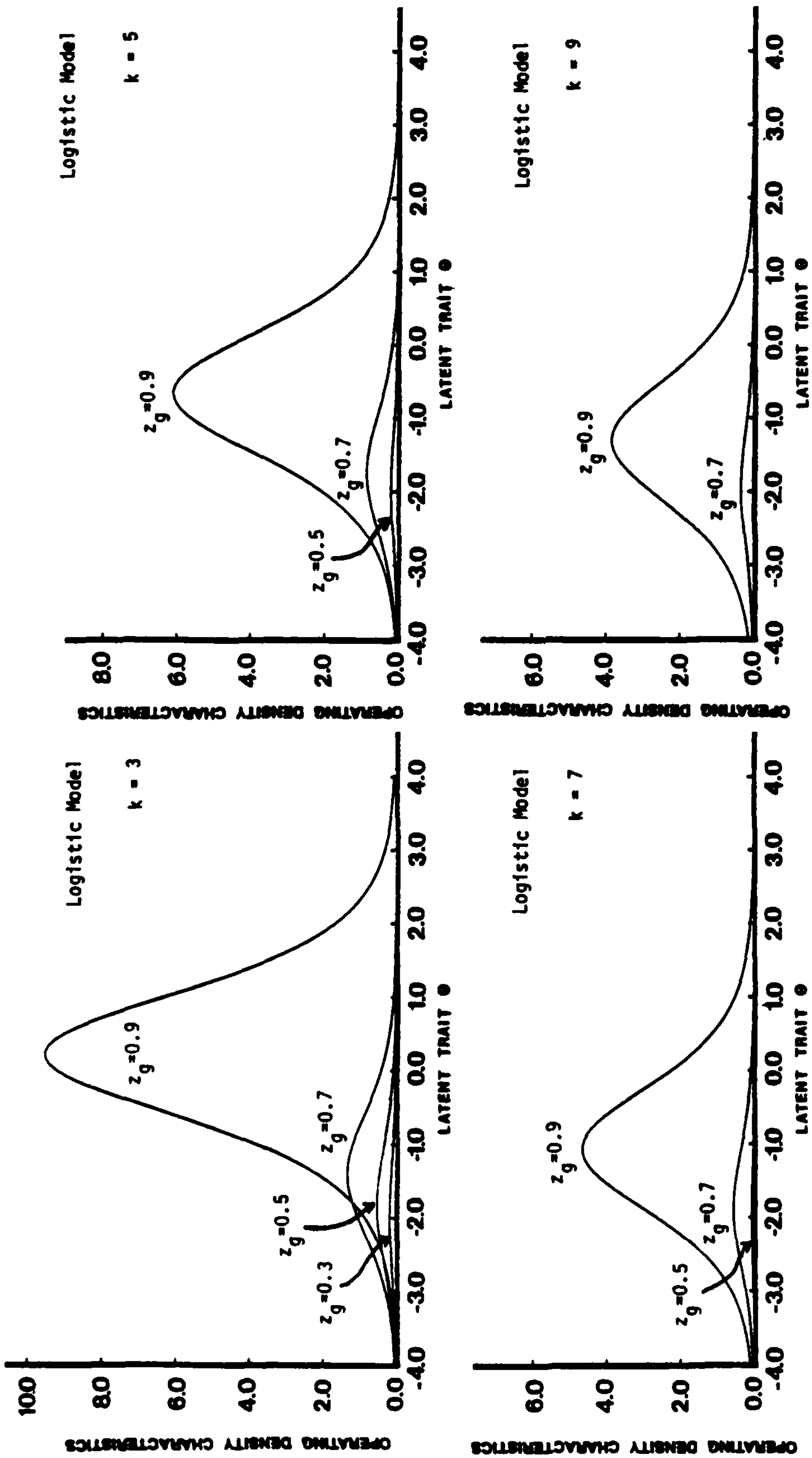

FIGURE A-7 (Continued)

APPENDIX VII

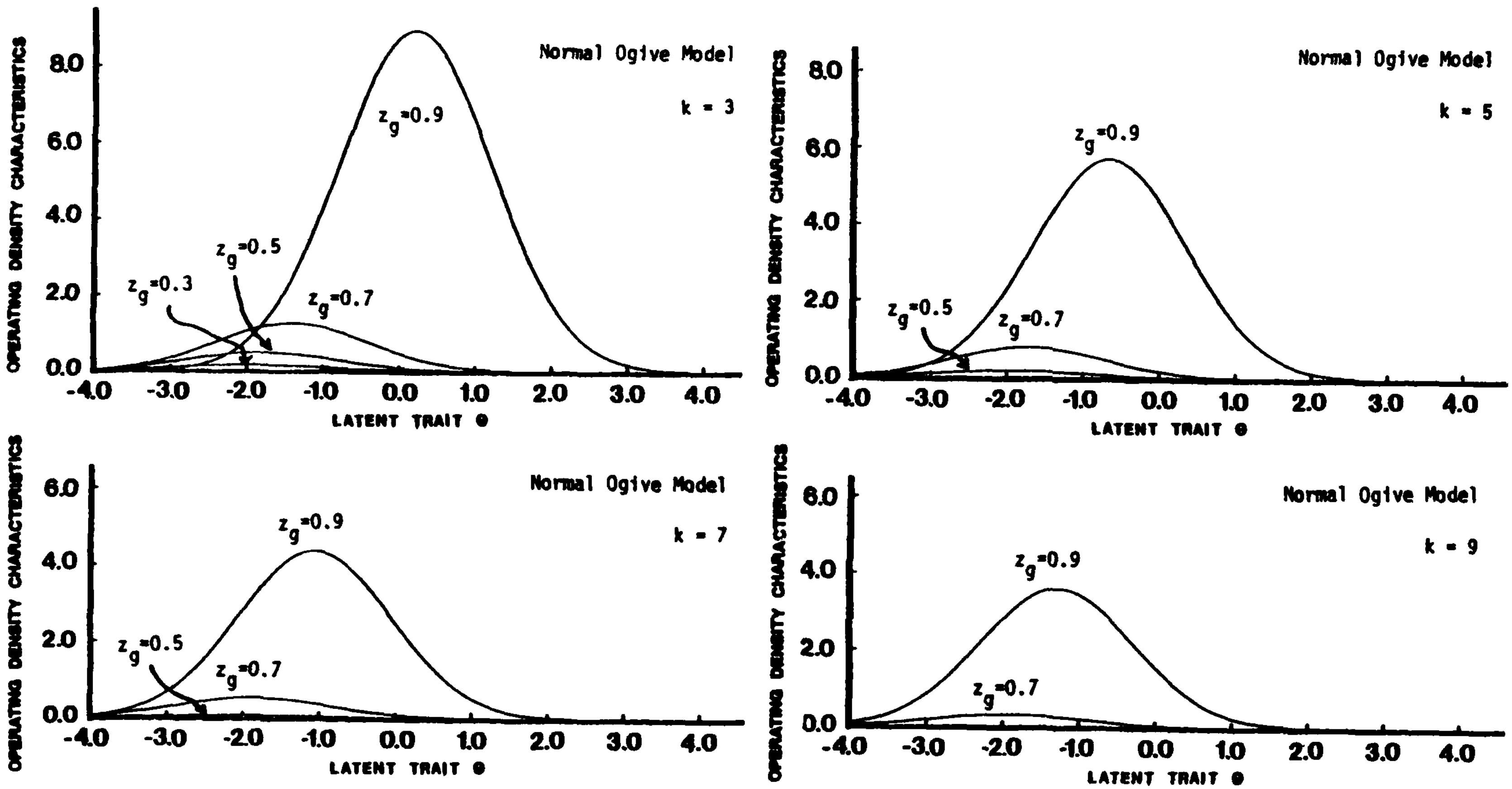

FIGURE A-7

Operating Density Characteristic, $H_{z_g}(\theta)$, As a Function of θ for Each of the Five Values of the Item Score, 0.1 , 0.3 , 0.5 , 0.7 and 0.9 , Following the Normal Ogive and the Logistic Models, with $a_g = 1.0$, $b_0 = -2.0$ and $D = 1.7$, When the Functional Relationship between the Item Score z_g and the Difficulty Parameter b_{z_g} Is Given by $b_{z_g} = b_0 + \tan[(\pi/2)z_g^k]$ $(k = 3, 5, 7, 9)$. The Additional Curve Is the One in the Limiting Situation Where z_g Tends to Zero. Closed/Open Response Situation.

APPENDIX VI

Logistic Model

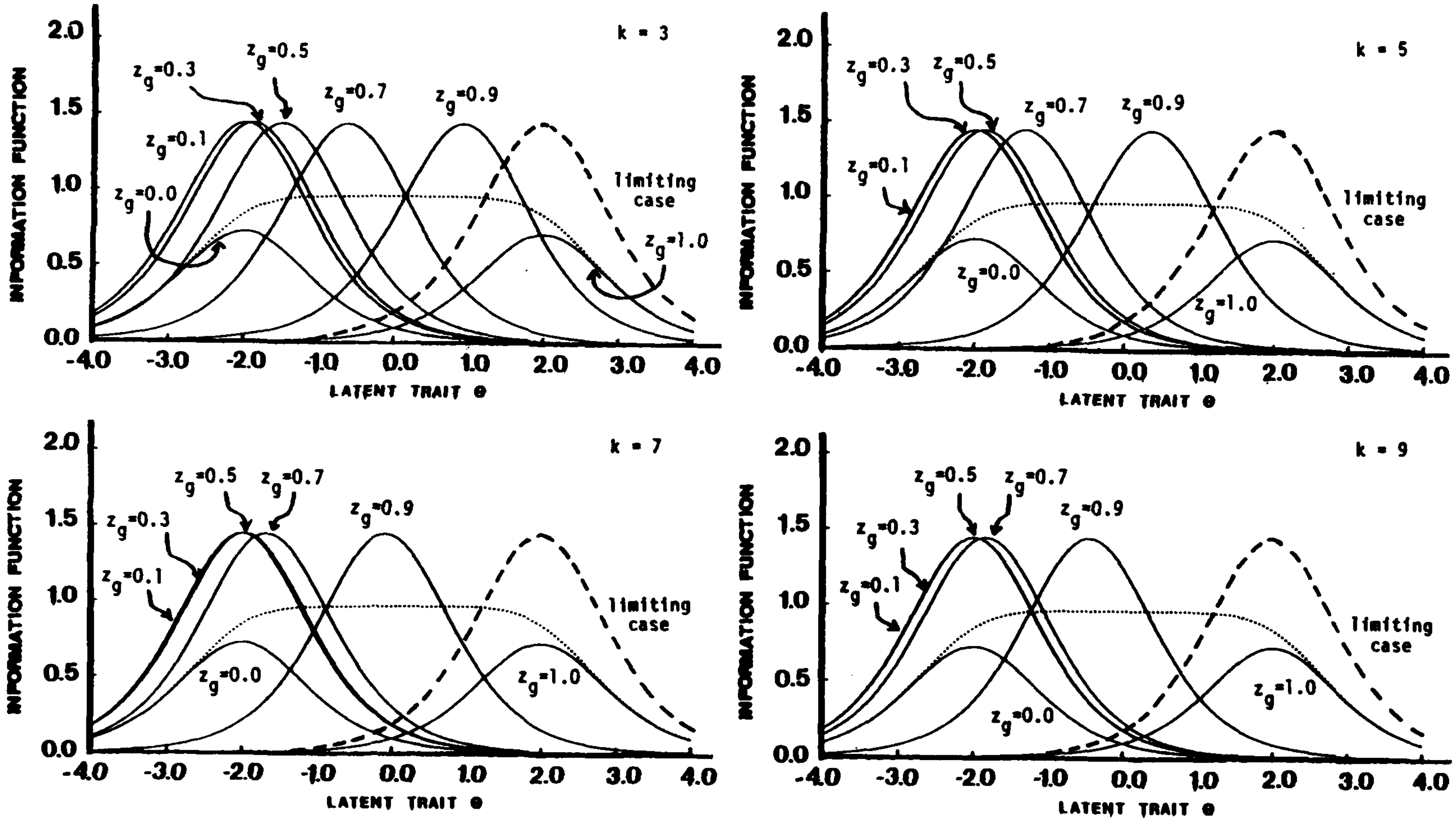

FIGURE A-6

Item Response Information Functions, $I_{z_g}(\theta)$, (Solid Line) and Item Information Function, $I_g(\theta)$, (Dotted Line) in the Logistic Model for $z_g = 0.0, 0.1, 0.3, 0.5, 0.7, 0.9, 1.0$, with $a_g = 1.0$, $b_0 = -2.0$, $b_1 = 2.0$ and $D = 1.7$, When the Functional Relationship between z_g and b_{z_g} Is Given by $b_{z_g} = b_0 + (b_1 - b_0) z_g^k$ for $k = 3, 5, 7, 9$. Closed Response Situation.

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